

State-space approach to zero-modules of proper transfer functions

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Abstract

The poles and zeros of a transfer function can be studied by various means. The main motivation of the present paper is to give a **state-space** description of the module theoretic definition of zeros introduced and analyzed by Wyman et al. in [15] and [16]. This analysis is carried out for proper transfer functions.

The obtained explicit equations determined by the system matrices are used for defining two *inner functions* to transform the original transfer function into a *square, invertible* one via multiplication eliminating the “generic” zeros corresponding to the kernel and the image of the transfer function.

As it is well-known the zeros are connected to various invariant subspaces arising in geometric control, see e.g. Aling and Schumacher [1] for a complete description. The connections to these subspaces are also mentioned in the paper.

Keywords: zeros of transfer function, realization of transfer function, state-space description, proper rational function, output-nulling subspace, controlled invariant subspace, input-containing subspace.

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1 Introduction

The study of zeros of transfer functions has already a long history. Various zeros has been defined, various approaches has been used to describe them. We are not brave enough to give a detailed description of this history but the book written by H. Rosenbrock ('70) should be cited here [12] as well as that of T. Kailath ('80) [6]. One of the approaches used in these books to define the zeros of a transfer function is based on the Smith-McMillan form of these functions. These are the so-called transmission zeros. C. B. Schrader and M. K.

Sain ('89) in [13] give a survey on the notions and results of zeros of linear time invariant systems, including invariant zeros, system zeros, input-decoupling zeros, output-decoupling zeros and input-output-decoupling zeros, as well. The connection of these zeros to invariant subspaces appearing in geometric control theory was considered e.g. in A. S. Morse ('73) [10] for strictly proper transfer functions, for proper transfer functions – not assuming the minimality of the realization – in H. Aling and J. M. Schumacher ('84) [1] showing that the combined decomposition of the state space considering Kalman's canonical decomposition and Morse's canonical decomposition in the same lattice diagram corresponds to the various notions of multivariate zeros.

The book written by J. Ball, I. Gohberg and L. Rodman [5] uses the concept of left (and Right) zero pairs. This offers the possibility of analyzing – together with the position of the zeros – the corresponding zero directions, as well.

The zeros play an important role in the theory of spectral factors. The connection between the zeros of spectral factors, splitting subspaces and the algebraic Riccati-inequality was studied in A. Lindquist et al. ('95) [7]. An important aspect of this paper was further analyzed by P. Fuhrmann and A. Gombani ('98) where the concept of *externalized* zeros was introduced. (Interestingly, this concept can be formulated in the framework of the dilation theory, as was pointed out by the author in [8].)

The starting point of the present paper is the module-theoretic approach to the zeros of multivariate transfer functions defined by B. F. Wyman and M. K. Sain ('83) [14], and further analyzed by Wyman et al. in [15], [16]. In this extension the so-called Wedderburn-Forney-spaces play an important role. (Although the published version of the paper written by G. D. Forney [3] does not contain an explicit definition of this construction, it was in the original manuscript.) The main result in [16] is that the number of zeros and poles of a rational transfer function coincide (even in the matrix case) assuming that the zeros are counted in a *right way*. It is well-known that to define the multiplicity of a finite zero (or even an infinite zero) the Rosenbrock matrix provides an appropriate tool. But it is an easy task to construct (non-square) matrix-valued transfer function with no finite (infinite) zeros. In such cases it might happen that there are rational functions mapped to the *identically* zero function by the transfer function. Then the functions in the kernel of the transfer function form an infinite dimensional vector space over the space of scalars, but it is finite dimensional over the field of rational functions. But defining the multiplicity of this *zero-function* as the corresponding dimension of the kernel subspace does not give a satisfactory result. To this aim the notion of minimal polynomial bases should be used as in [3] by G. D. Forney.

The main motivation of the present paper is to give a matrix theoretic description of the corresponding zero-concepts, i.e. to show how to compute these zero-modules starting from a state-space realization of the transfer function.

Section 2 gives a short introduction to the zero-modules and minimal polynomial bases.

Section 3 first refreshes the fact that the finite zeros can be described by the Rosenbrock-matrix, namely if $F(z) = D + C(zI - A)^{-1}B$ then the equation
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi \\ H \end{bmatrix} = \begin{bmatrix} \Pi A \\ 0 \end{bmatrix}$$
 should be considered. (The minimality of the realization will not be assumed in the paper, only the *observability* of the pair (C, A) .) But it turns out that the same equation describes the zeros corresponding to the kernel-module, as well, and although there is a possibility

to consider a *maximal* solution of this equation, this maximality is well-defined in terms of $\text{Im}(\Pi)$ but in general the matrix Λ (and H) is not uniquely defined. Loosely speaking, some part of it can be freely chosen. It is shown that for this maximal solution the subspace $\text{Im}(\Pi)$ is the maximal output-nulling controlled invariant subspace (denoted by $\mathcal{V}^*(\Sigma)$, where Σ indicates the system), while the maximal output-nulling reachability subspace (denoted by $\mathcal{R}^*(\Sigma)$) describes that part of the matrix Λ where it is not uniquely defined by the system matrices. A maximal solution of the equation $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ R_0 \end{bmatrix} = \begin{bmatrix} \Pi\alpha_0 \\ 0 \end{bmatrix}$ should be considered and $\mathcal{R}^*(\Sigma)$ is given $\text{Im}(\Pi \langle \Lambda \mid \alpha_0 \rangle)$, where $\langle \Lambda \mid \alpha_0 \rangle$ denotes the minimal Λ -invariant subspace containing $\text{Im}(\alpha_0)$. As a side result, we obtain that the minimal indices corresponding to the kernel of F coincide with the controllability indices of the pair (Λ, α_0) . It should be noted here that the correspondence between the various zeros and the various invariant subspaces was thoroughly investigated e.g. in [1] by Aling and Schumacher even in the general non-minimal case. Especially, they proved that $\mathcal{R}^*(\Sigma)$ corresponds to the kernel of F , while $(\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle) / \mathcal{R}^*(\Sigma)$ to the finite transmission zeros (assuming the observability of (C, A)). But the explicit reference to the equation above was not given by them.

Using the maximal solutions of the equations above a matrix valued tall inner (in continuous time sense) function $K(z)$ is constructed explicitly with columns forming a basis (over the field of rational functions) in the kernel of the transfer function. Via a square-inner extension L of K (i.e. $[K, L]$ is a square inner function) the generic zeros corresponding to the kernel-zero module can be turned into finite zeros, in other words the function $F_{\mathbf{r}} = FL$ has already a trivial kernel (moreover it is left-invertible) but still containing the original finite zeros of F . In terms of the language of geometric control theory, starting from a minimal realization of F and deriving from this a realization for $F_{\mathbf{r}}$ these realizations share the same maximal output-nulling controlled invariant subspace: $\mathcal{V}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}^*(\Sigma)$, but $\mathcal{R}^*(\Sigma_{\mathbf{r}})$ becomes trivial.

In order to eliminate the defect in the image space in Section 4 first the connection between the left and right zero-modules is analyzed showing especially that if for the same transfer function the roles of the input signal and the output signal are changed (i.e. instead of the effect of the right multiplication $g \rightarrow Fg$ the left multiplication $h \rightarrow hF$ is considered) then – assuming a minimal realization is taken – the orthogonal complement of the maximal output-nulling controlled invariant subspace defined for right multiplication is the minimal input-containing subspace defined for the left multiplication. Shortly, $(\mathcal{V}^*(\Sigma))^{\perp} = \mathcal{C}^*(\Sigma)_{\text{left}}$.

Now using an appropriate flat inner function L' the left kernel-zero module of F can be eliminated. Simultaneous application of the inner functions L and L' leads to the following definition: $F_{\mathbf{rl}} = L'FL$. Theorem 4.5 claims that if the poles of F are in the closed left half-plane while there is no finite zero on the imaginary axis then the McMillan degree of $F_{\mathbf{rl}}$ is the same as that of F , and the function $F_{\mathbf{rl}}$ has only finite and possibly infinite zeros, thus its kernel-zero module and the zero module corresponding to the defect in the image space are trivial. The inner functions L' and L transform the so-called generic zeros into finite zeros positioned in the open right half-plane. The function $F_{\mathbf{rl}}$ is a square, invertible function, thus the "squaring" of F is achieved via left- and right multiplication, preserving the poles of the original transfer function. In the paper written by Ntogramatzidis and Prattichizzo

[11] this squaring is obtained via state-feedback and output-injection.

2 Preliminaries and notation

Let U and Y be vector spaces over \mathbb{C} of dimensions q and p , respectively. As usual, $\mathbb{C}(z)$ denotes the field of rational functions, $\mathbb{C}[z]$ the ring of polynomials over \mathbb{C} . Set

$$U(z) = U \otimes_{\mathbb{C}} \mathbb{C}(z), \quad Y(z) = Y \otimes_{\mathbb{C}} \mathbb{C}(z).$$

(these are the sets of vector valued rational functions).

Let $F(z)$ be a *transfer function*, i.e. an $\mathbb{C}(z)$ linear map

$$F(z) : U(z) \rightarrow Y(z).$$

Choosing bases (over \mathbb{C}) in U and Y we obtain bases for $U(z)$ and $Y(z)$ (over $\mathbb{C}(z)$) and a $p \times q$ matrix representation for $F(z)$.

Let us introduce the notations

$$\Omega U = U \otimes_{\mathbb{C}} \mathbb{C}[z], \quad \Omega Y = Y \otimes_{\mathbb{C}} \mathbb{C}[z],$$

(these are the sets of vector-valued polynomials) and

$$\Omega_{\infty} U = U \otimes_{\mathbb{C}} \mathcal{O}_{\infty}, \quad \Omega_{\infty} Y = Y \otimes_{\mathbb{C}} \mathcal{O}_{\infty},$$

where \mathcal{O}_{∞} denotes the set of proper rational functions in $\mathbb{C}(z)$. Obviously,

$$z^{-1}\Omega_{\infty} U, \quad z^{-1}\Omega_{\infty} Y$$

are the sets of the strictly proper vector-valued rational functions.

Following R. Kalman we might identify the set ΩU with the (finite) past (with respect to the zero time point) inputs, and ΩY with the (finite) past outputs.

2.1 Zero and pole modules of a transfer function

In this subsection we recall the definition of the pole and zero modules following Wyman and Sain [14]. The finite pole module is given as

$$X(F) = \frac{\Omega U}{F^{-1}(\Omega Y) \cap \Omega U}.$$

That is, the set of polynomial inputs is factorized by the polynomial inputs giving rise to polynomial outputs.

Similarly, the infinite pole module is

$$X_{\infty}(F) = \frac{z^{-1}\Omega_{\infty} U}{F^{-1}(z^{-1}\Omega_{\infty} Y) \cap z^{-1}\Omega_{\infty} U}.$$

To define the zero module we might start with

$$\frac{F^{-1}(\Omega Y)}{F^{-1}(\Omega Y) \cap \Omega U} ,$$

(the set of inputs leading to polynomial outputs factorized by the inputs which are themselves polynomial, in other words the set of inputs producing no outputs after time zero where two inputs are considered to be equivalent if they differ only in the past).

In those cases, when there are inputs producing identically zero outputs, in other words the kernel of the transfer function is nontrivial, then the space above is infinite dimensional (over \mathbb{C}). Factorizing out this kernel we obtain the "module of finite zeros":

$$\begin{aligned} Z(F) &= \left(\frac{F^{-1}(\Omega Y)}{F^{-1}(\Omega Y) \cap \Omega U} \right) / \left(\frac{\ker F(z)}{\ker F(z) \cap \Omega U} \right) \\ &= \frac{F^{-1}(\Omega Y) + \Omega U}{\ker F + \Omega U} . \end{aligned}$$

The infinite zero module is defined similarly

$$Z_{\infty}(F) = \frac{F^{-1}(\Omega_{\infty} Y) + \Omega_{\infty} U}{\ker F + \Omega_{\infty} U} ,$$

To define a finite-dimensional object *counting* the "number of zeros" corresponding to the possibly infinite dimensional (over \mathbb{C}) of $\ker F$ there are two possibilities offered by Forney [3]. The first one is based on the so-called Wedderburn-Forney spaces, the second one uses the notion of "minimal polynomial bases".

To define the first one let us start with introducing a mapping π_- rendering to any rational function its strictly proper part. I.e

$$\pi_- : \mathbb{C}(z) \rightarrow z^{-1}\mathcal{O}_{\infty} .$$

This can be extended in an obvious manner to a mapping from $U(z)$ to $z^{-1}\Omega_{\infty}U$, and also to a mapping from $Y(z)$ to $z^{-1}\Omega_{\infty}Y$. Both these extended mappings will be denoted by the same symbol π_- . (Similarly, π_+ denotes the mapping producing the polynomial part of any rational function.)

Now the kernel subspace $\ker F$ is obviously a module over $\mathbb{C}(z)$. The Wedderburn-Forney space obtained from it is denoted by $\mathcal{W}(\ker F)$ and defined as

$$\mathcal{W}(\ker F) = \frac{\pi_-(\ker F)}{\ker F \cap z^{-1}\Omega_{\infty}U}$$

According to Theorem 5.1 (and Corollary 5.2) in Wyman et al. [15] for every rational transfer function the number of poles and zeros are equal, if they are "counted" in an appropriate way. Namely, set

$$\mathcal{X}(F) = X(F) \oplus X_{\infty}(F) , \quad \mathcal{Z}(F) = Z(F) \oplus Z_{\infty}(F) .$$

Then for some linear mappings α and β the sequence

$$0 \rightarrow \mathcal{Z}(F) \xrightarrow{\alpha} \frac{\mathcal{X}(F)}{\mathcal{W}(\ker F)} \xrightarrow{\beta} \mathcal{W}(\operatorname{Im} F) \rightarrow 0$$

forms an exact sequence. The mapping α is induced by the mapping

$$(u, v) \rightarrow (\pi_+(u + v), \pi_-(u + v)) ,$$

from $F^{-1}(\Omega Y) \oplus F^{-1}(z^{-1}\Omega_\infty Y) \rightarrow \Omega U \oplus z^{-1}\Omega_\infty U$, while β is induced by

$$(u, v) \rightarrow \pi_- [F \cdot (u + v)]$$

from $\Omega U \oplus z^{-1}\Omega_\infty U \rightarrow \pi_- \operatorname{Im} F$. (More precisely, to define the factor space $\frac{\mathcal{X}(F)}{\mathcal{W}(\ker F)}$ first we have to apply the imbedding of $\mathcal{W}(\ker F)$ into $\mathcal{X}(F)$ induced by the linear mapping $\pi_- u \rightarrow (\pi_+ u, \pi_- u)$ from $\ker F \rightarrow \Omega U \oplus z^{-1}\Omega_\infty U$.)

2.2 Minimal polynomial basis

Now let us turn to the second possibility based on the notion of minimal polynomial basis.

If $v = (v_1, \dots, v_k)$ is a k -tuple of polynomials, then set $\deg v = \max_j \deg v_j$. If V is a $k \times m$ array of polynomials, then its (column)-degree is $\nu = \sum_l \nu_l$, where $V = [v_1, \dots, v_m]$, i.e. v_l denotes the l -th column of V , and $\nu_l = \deg v_l$. Denote by V_h the matrix of highest (column) degree coefficients of V .

Definition 2.1 (Minimal basis). *Let \mathcal{V} be a finite dimensional subspace of k -tuples over the field of rational functions $\mathbb{C}(z)$. The array V is a minimal (polynomial) basis of \mathcal{V} (over the rational functions), if*

- *it has polynomial entries,*
- *its columns form a basis over the rational functions of \mathcal{V} , and*
- *has least (column)-degree.*

Denote by $\deg_{\min} \mathcal{V}$ the degree of any minimal polynomial basis in \mathcal{V} .

Then according to Corollary 6.5 in Wyman et al. [15]

$$\dim \mathcal{W}(\ker F) = \deg_{\min} \ker F .$$

Similarly,

$$\dim \mathcal{W}(\operatorname{Im} F) = \deg_{\min} \operatorname{Im} F .$$

Thus the correct formulation the statement about the number of zeros and poles is as follows:

$$\dim \mathcal{X}(F) = \dim \mathcal{Z}(F) + \deg_{\min} \ker F + \deg_{\min} \operatorname{Im} F .$$

Later on we need the following characterization of minimal (polynomial) bases given by Forney [3].

Theorem 2.1. *Let \mathcal{V} be an n -dimensional subspace of k -tuples of rational functions. Assume that $V = [v_1, \dots, v_m]$ has polynomial entries. Then the following properties are equivalent:*

- (i) V is a minimal basis of \mathcal{V} ;
- (ii) if $\xi = V\zeta$ is a polynomial k -tuple, $\xi \in \mathcal{V}$, then ζ must be a polynomial m -tuple, and $\deg \xi = \max_{1 \leq l \leq m} [\deg \zeta_l + \nu_l]$;
- (iii) $\dim \mathcal{V}_d = \sum_l (d - \nu_l)^+$, where \mathcal{V}_d denotes the set of polynomials in \mathcal{V} of degree strictly less, then d ,
- (iv) for any complex number z_0 the matrix $V(z_0)$ has full column rank, and V_h is also of full column rank.

2.3 State-space realization, invariant subspaces

In this paper we are going to characterize the zero modules of a proper transfer function $F(z)$ based on some linear equations using the so-called Rosenbrock-matrix associated to F .

Following Rosenbrock we shall use the notation

$$F(z) \sim \Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

indicating that $F(z) = D + C(zI - A)^{-1}B$.

Although the definition of zeros considered in this paper of a rational function does not depend on whether a continuous or a discrete time system is associated to it it will turn out of the analysis later that discrete time systems arise in a natural way. Namely, the system

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) , \\ y(k) &= Cx(k) + Du(k) . \end{cases} \quad (2.1)$$

The subspace $\langle A \mid B \rangle = \text{Im} [B, AB, A^2B, \dots]$ is the reachability subspace of the state-space.

A subspace \mathcal{V} of the state space is called **output-nulling controlled-invariant** if there exists a feedback map K such that

$$(A + BK)\mathcal{V} \subset \mathcal{V} \subset \ker(C + DK) .$$

It is well-known that there exists a maximal output-nulling controlled-invariant set – denoted by $\mathcal{V}^*(\Sigma) = \mathcal{V}^*(A, B, C, D)$. (See for example P. A. Fuhrmann and U. Helmke [4] where these sets are characterized using polynomial and rational models of state-space systems.) Note that $\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle$ is also an output-nulling controlled-invariant subspace.

The set of the **output-nulling reachable** elements $\mathcal{C}^*(\Sigma)$ of the system (2.1) also plays important role in this paper. This is defined as follows:

$$\mathcal{C}^*(\Sigma) = \left\{ x \mid \begin{array}{l} \exists (\dots, 0, u(-k), u(-k+1), \dots, u(0)) \text{ input such that} \\ y(j) = 0, j \leq 0 \quad \text{and} \quad x = x(1) \end{array} \right\} \quad (2.2)$$

Obviously $\mathcal{C}^*(\Sigma) = \mathcal{C}^*(A, B, C, D)$ forms a subspace of the state-space. Note that $\mathcal{C}^*(\Sigma)$ coincides with the **minimal input-containing** subspace. See e.g. Aling and Schumacher [1]. (A subspace \mathcal{C} is called input containing if there exists an output-injection L such that $(A + LC)\mathcal{C} \subset \mathcal{C}$ and $\text{Im}(B + LD) \subset \mathcal{C}$.)

The intersection $\mathcal{R}^*(\Sigma) = \mathcal{V}^*(\Sigma) \cap \mathcal{C}^*(\Sigma)$ is the **maximal output-nulling reachability** subspace. (See again [1].)

For a matrix A its adjoint will be denoted by A^* , while for a matrix valued function $F(z)$ the notation $F^*(z)$ refers to its para-hermitian conjugate function, i.e. $F^*(z) = (F(-\bar{z}))^*$.

3 Zeros of proper transfer functions

As we have seen there are several ingredients of the “zero structure” of a transfer function.

Let us recall that to determine the finite zero module $Z(F)$ first we have to characterize the set $F^{-1}(\Omega Y) + \Omega U$, i.e. those functions h for which there exists a polynomial q -tuple ψ (in ΩU) such that $\phi = F \cdot (h + \psi)$ is a polynomial p -tuple. In order to get $Z(F)$ this should be factorized with respect to $\ker F + \Omega_\infty U$. This set contains the functions h for which there exists a polynomial q -tuple ψ such that $F \cdot (h + \psi) = 0$.

Similarly, to characterize the infinite zero module we have to consider $F^{-1}(\Omega_\infty Y) + \Omega_\infty U$ with a similar characterization as above but instead of polynomials we have to consider proper functions. To get $Z_\infty(F)$ this should be factorized with respect to $\ker F + \Omega_\infty U$.

To obtain the kernel-module $\mathcal{W}(\ker F)$ we have to first consider $\pi_-(\ker F)$, i.e. the set of those strictly proper functions h for which there exists a polynomial q -tuple ψ such that $F \cdot (h + \psi) = 0$. In order to get $\mathcal{W}(\ker F)$ this set should be factorized with respect to the set of strictly proper functions in the kernel of F .

Similarly to describe $\mathcal{W}(\text{Im} F)$ we should first consider $\pi_-(\text{Im} F)$, i.e. the strictly proper functions h for which there exists a polynomial p -tuple ϕ such that $h + \phi \in \text{Im}(F)$. Two functions h_1, h_2 are considered to be equivalent if $h_1 - h_2 \in \text{Im}(F)$.

3.1 The finite zero-module $Z(F)$ and the kernel-module $\mathcal{W}(\ker F)$

Let us start with the analysis of $F^{-1}(\Omega Y) + \Omega U$ appearing in the definition of $Z(F)$.

Let us point out that Theorem 1 in Michaletzky-Gombani [9] essentially characterizes these functions.

Theorem 3.1. *Let $F(z) = D + C(zI - A)^{-1}B$ be a rational function.*

(i) *Assume that there exists a - possibly matrix-valued - function*

$$g(z) = H(zI - \Lambda)^{-1}G + \psi(z),$$

where ψ is a matrix-valued polynomial and (Λ, G) is a controllable pair such that

Fg is analytic at the eigenvalues of Λ .

If moreover the pair (C, A) is observable, then there exists a matrix Π such that $\text{Im } \Pi \subset \langle A \mid B \rangle$ solving the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi \\ H \end{bmatrix} = \begin{bmatrix} \Pi \Lambda \\ 0 \end{bmatrix}. \quad (3.3)$$

(ii) Assume that the matrices Λ, H, Π satisfy the equation (3.3), where (H, Λ) is an observable pair and $\text{Im } \Pi \subset \langle A \mid B \rangle$. Then there exists a matrix polynomial ψ such that for

$$g(z) = H(zI - \Lambda)^{-1} + \psi(z) \quad (3.4)$$

the function Fg is analytic at the eigenvalues of Λ .

Finally, equation (3.3) implies that

$$F(z)H(zI - \Lambda)^{-1} = -C(zI - A)^{-1}\Pi, \quad (3.5)$$

which is already analytic at the eigenvalues of Λ if A and Λ have no common eigenvalues.

Furthermore, in part (ii) the polynomial ψ can be chosen in such a way that the product Fg be a polynomial. In other words, the columns of the function $H(zI - \Lambda)^{-1}$ are in $F^{-1}(\Omega Y) + \Omega U$.

This gives the possibility of formulating Theorem 3.1 in the following way.

Theorem 3.2. Let $F(z) = D + C(zI - A)^{-1}B$ be a rational function.

(i) Assume that the pair (C, A) is observable and the pair (Λ, G) is controllable. Then, if the columns of the function

$$g(z) = H(zI - \Lambda)^{-1}G,$$

are in $F^{-1}(\Omega Y) + \Omega U$, then there exists a matrix Π such that $\text{Im } \Pi \subset \langle A \mid B \rangle$ solving the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi \\ H \end{bmatrix} = \begin{bmatrix} \Pi \Lambda \\ 0 \end{bmatrix}. \quad (3.6)$$

(ii) Assume that the matrices Λ, H, Π satisfy the equation (3.6), where (H, Λ) is an observable pair and $\text{Im } \Pi \subset \langle A \mid B \rangle$. Then there exists a matrix polynomial ψ such that for

$$g(z) = H(zI - \Lambda)^{-1} + \psi(z) \quad (3.7)$$

the function Fg is a polynomial, i.e. the columns of the matrix-valued rational function $H(zI - \Lambda)^{-1}$ are in $F^{-1}(\Omega Y) + \Omega U$.

Corollary 3.1. Assume that $F(z) = D + C(zI - A)^{-1}B$ is a minimal realization, and (H, Λ) is an observable pair.

Then the columns of $H(zI - \Lambda)^{-1}$ are in the set $F^{-1}(\Omega Y) + \Omega U$ if and only if equation (3.6) holds.

Later on we shall utilize the following proposition which in a sense can be considered as a converse of the last statement in Theorem 3.1.

Proposition 3.1. *Assume that equation*

$$[D + C(zI - A)^{-1}B] [H(zI - \Lambda)^{-1}G] = -C(zI - A)^{-1}S \quad (3.8)$$

holds.

Then if the pair (C, A) is observable and the pair (Λ, G) is controllable then there exists a matrix Π such that equation (3.6) holds, as well. Moreover

$$\Pi G = S.$$

PROOF. Let us observe that if the matrices A and Λ have no common eigenvalues then equation (3.8) implies that the product is analytic at the eigenvalues of Λ , thus Theorem 3.1 (i) implies immediately that equation (3.3) holds true.

In the general case let us observe that equation (3.8) can be written in the following form, as well.

$$[C, DH] \left(z \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A & BH \\ 0 & \Lambda \end{bmatrix} \right)^{-1} \begin{bmatrix} S \\ G \end{bmatrix} = 0 \quad (3.9)$$

In other notation

$$0 \sim \left(\begin{array}{cc|c} A & BH & S \\ 0 & \Lambda & G \\ \hline C & DH & 0 \end{array} \right).$$

Let us first consider the unobservability subspace of the realization obtained above of the identically zero function. Suppose that the columns of the matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ form a basis in the unobservability subspace. Then

$$[C \quad DH] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0, \quad \begin{bmatrix} A & BH \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \rho,$$

for some matrix ρ .

If for some vector ξ the product $\beta\xi = 0$ then $\beta\rho\xi = 0$, as well. Thus the subspace $\ker \beta$ is ρ -invariant. Consider now an eigenvector ξ of ρ belonging to this subspace. Then

$$\rho\xi = \lambda\xi, \quad C\alpha\xi = 0, \quad A\alpha\xi = \alpha\rho\xi = \lambda\alpha\xi.$$

The observability of the pair (C, A) implies that $\alpha\xi = 0$, as well, thus the columns of the matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are linearly dependent, contrary to our assumption. Consequently, the columns of β are linearly independent. Equation $\Lambda\beta = \beta\rho$ implies that dimension of the unobservability subspace cannot be greater than the size of the matrix Λ .

Now let us assume that the row vectors of the matrix $[\gamma, \delta]$ form a basis in the orthogonal complement of the controllability subspace. Then

$$[\gamma \quad \delta] \begin{bmatrix} S \\ G \end{bmatrix} = 0, \quad [\gamma \quad \delta] \begin{bmatrix} A & BH \\ 0 & \Lambda \end{bmatrix} = \tau [\gamma \quad \delta].$$

Using the controllability of the pair (Λ, G) similar reasoning shows that the codimension of the controllability subspace cannot be larger than the size of A . Since these two subspaces together should generate the whole space we obtain that – comparing the dimensions of these subspaces with the sizes of the corresponding matrices – equalities should hold. Thus β and γ should be square matrices with trivial kernels. Applying nonsingular transformations it can be assumed that both are identity matrices. Then especially

$$\tau = A, \quad S + \delta G = 0, \quad BH + \delta \Lambda = A\delta.$$

Substituting the equation $BH = (zI - \Lambda)\delta - \delta(zI - A)$ into (3.8) straightforward computation gives that

$$(DH - C\delta)(zI - \Lambda)^{-1}G = 0.$$

The controllability of (Λ, G) implies that $DH - C\delta = 0$. Thus the matrix $\Pi = -\delta$ satisfies the required equations. \blacksquare

Note that equation (3.6) implies that there exists a state feedback K such that

$$(A + BK)\text{Im}\Pi \subset \text{Im}\Pi \subset \ker(C + DK).$$

Thus the columns of Π are in the maximal output-nulling controlled invariant set of the system (2.1). I.e.

$$\text{Im}(\Pi) \subset \mathcal{V}^*(\Sigma).$$

For the sake of completeness we provide a proof of the next obvious statement showing that $\mathcal{V}^*(\Sigma) = \mathcal{V}^*(A, B, C, D)$ can be characterized via the "maximal solution" of equation (3.6).

Lemma 3.1. *For any system Σ determined by the matrices A, B, C, D there exists a maximal solution $(\Pi_{max}, H_{max}, \Lambda_{max})$ of the equation*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{max} \\ H_{max} \end{bmatrix} = \begin{bmatrix} \Pi_{max}\Lambda_{max} \\ 0 \end{bmatrix} \quad (3.10)$$

in the sense that

$$\text{Im}(\Pi_{max}) \supset \text{Im}(\Pi_1)$$

if (Π_1, H_1, Λ_1) is any other solution of the equation above.

Moreover, for this maximal solution

$$\mathcal{V}^*(\Sigma) = \mathcal{V}^*(A, B, C, D) = \text{Im}(\Pi_{max}).$$

PROOF. If (Π_1, H_1, Λ_1) and (Π_2, H_2, Λ_2) are solutions of the equation then

$$\left([\Pi_1, \Pi_2], [H_1, H_2], \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \right)$$

is a solution, as well. Since

$$\text{Im}[\Pi_1, \Pi_2] = \text{Im}\Pi_1 \vee \text{Im}\Pi_2$$

the subspace generated by the ranges of all solutions is also the range of a solution, proving that there exist a maximal solution.

To prove the last statement let us point out that we have already observed that $\text{Im}(\Pi) \subset \mathcal{V}^*(\Sigma)$ for any solution (Π, H, Λ) of (3.6).

For the converse inclusion consider a matrix Π^* with column vectors forming a basis in $\mathcal{V}^*(\Sigma)$. Then there exists a feedback matrix K^* such that the inclusions

$$(A + BK^*) \text{Im}\Pi^* \subset \text{Im}\Pi^* \subset \ker(C + DK^*)$$

hold. In other words

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi^* \\ K^* \Pi^* \end{bmatrix} = \begin{bmatrix} \Pi^* \Lambda^* \\ 0 \end{bmatrix}$$

for some matrix Λ^* .

The first part of the lemma implies that $\mathcal{V}^*(\Sigma) = \text{Im}\Pi^* \subset \text{Im}\Pi_{\max}$, if $(\Pi_{\max}, H_{\max}, \Lambda_{\max})$ is a maximal solution, concluding the proof of the lemma. \blacksquare

Later on we need the following simple property of the subspace $\text{Im}(\Pi_{\max})$.

Proposition 3.2. *If*

$$x_+ = Ax + Bu \tag{3.11}$$

$$0 = Cx + Bu \tag{3.12}$$

and $x_+ \in \text{Im}(\Pi_{\max})$ then $x \in \text{Im}(\Pi_{\max})$, as well, where Π_{\max} is a maximal solution of equation (3.10).

PROOF. Equations (3.11) and (3.10) together can be written as follows:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{\max} & x \\ H_{\max} & u \end{bmatrix} = \begin{bmatrix} \Pi_{\max} \Lambda_{\max} & x_+ \\ 0 & 0 \end{bmatrix}. \tag{3.13}$$

According to the assumption there exists a vector ζ such that $x_+ = \Pi_{\text{zero}} \zeta$ giving the identity

$$\begin{bmatrix} \Pi_{\max} \Lambda_{\max} & x_+ \\ 0 & 0 \end{bmatrix} = [\Pi_{\max}, x] \begin{bmatrix} \Lambda_{\max} & \zeta \\ 0 & 0 \end{bmatrix}.$$

Substituting this into the left hand side of (3.13) and using the maximality of $\text{Im}(\Pi_{\max})$ we obtain that $x \in \text{Im}(\Pi_{\max})$. \blacksquare

Now let us return to the analysis of the space $F^{-1}(\Omega Y) + \Omega U$. Theorem 3.2 suggests that to describe the functions in this space we have to consider a maximal – in some sense – solution of equation (3.6). Lemma 3.1 shows that in terms of $\text{Im}(\Pi)$ there exists a maximal solution. But the following lemma shows that in terms of the triplet (Π, H, Λ) – in general – this is not possible.

Lemma 3.2. *Let h be a polynomial q -tuple. Assume that $Fh = 0$. Consider an arbitrary complex number $a \in \mathbb{C}$, and write $h(z) = \sum_{j=0}^k h_j(z-a)^j$. Set*

$$H = [h_0, h_1, \dots, h_{k-1}, h_k]$$

and

$$\Lambda_a = \begin{bmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & & 0 \\ 0 & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a & 1 \\ 0 & 0 & \cdots & 0 & a \end{bmatrix},$$

the Jordan-matrix corresponding to the value a .

Then – assuming the observability of the pair (C, A) – the function $g(z) = (zI - A)^{-1} Bh$ is also a polynomial, $g(z) = \sum_{j=0}^{k-1} g_j(z-a)^j$ and equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} G\Lambda_a \\ 0 \end{bmatrix} \quad (3.14)$$

holds where

$$G = [g_0, g_1, \dots, g_{k-1}, 0] .$$

PROOF. Set $g(z) = (zI - A)^{-1} Bh(z)$. Invoking Lemma 7.1 in [1] we get that g is a polynomial of degree $k-1$.

Arranging the identity $((z-a)I - (A - aI))g(z) = Bh(z)$ on the coefficients of h and g into matrix form we arrive at the equation (3.14) ■

Note that according [1] the following more general statement holds, as well. Consider a set of polynomials h_1, h_2, \dots, h_l from $\ker F$. Define $x_i(z) = (zI - A)^{-1} Bh_i(z)$, $j = 1, \dots, l$. According to the previous statement – under the assumed observability of the pair (C, A) – they are also polynomial. Then h_1, h_2, \dots, h_l form a minimal polynomial basis in $\ker F$ if and only if $\begin{bmatrix} x_1 \\ h_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ h_2 \end{bmatrix}, \dots, \begin{bmatrix} x_l \\ h_l \end{bmatrix}$ form a minimal polynomial basis in $\ker \begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix}$.

Since if there exists a rational function in the kernel of F then there is also a polynomial in it (multiplying with the common denominator of its entries), the previous lemma shows that in this case there is no *largest* matrix Λ containing all “zeros” of F as its eigenvalues. But to characterize the finite zero module $Z(F)$ only the equivalence classes in $F^{-1}(\Omega Y) + \Omega U$ should be taken, where two functions g_1, g_2 in it are considered to be equivalent if $g_1 - g_2 \in \ker F + \Omega U$. Especially the polynomial part can be eliminated. In other words,

$$g - \pi_-(g) \in \Omega U \subset \ker F + \Omega U .$$

and

$$g \in F^{-1}(\Omega Y) + \Omega U \quad \text{if and only if} \quad \pi_-(g) \in F^{-1}(\Omega Y) + \Omega U .$$

Thus our next goal is to characterize the *equivalence classes* in $F^{-1}(\Omega Y) + \Omega U$ via the equation (3.6). As we have seen a polynomial q -tuple in $\ker F$ also induces a solution of this equation. So what we might hope is that this equation is appropriate for characterizing not only the elements of $Z(F)$ but also including $\ker(F)$ or more precisely the elements in $\mathcal{W}(\ker F)$.

Let us observe that according to the definition of $\mathcal{W}(\ker F)$ a q -tuple $g \in \mathcal{W}(\ker F)$ if and only if there exists a polynomial ψ such that $g + \psi \in \ker F$, and two functions g_1, g_2 with this property are considered to be equivalent if $g_1 - g_2 \in \ker F \cap z^{-1}\Omega_\infty U$.

Summarizing these considerations to describe the elements of $Z(F) \oplus \mathcal{W}(\ker F)$ those rational q -tuples g should be considered for which there exists a polynomial q -tuple ψ such that $F(g + \psi)$ is a polynomial, and two functions g_1, g_2 are taken to be equivalent if and only if $g_1 - g_2 \in (\ker F) \cap z^{-1}\Omega_\infty U$. Obviously every equivalence class contains a strictly proper rational function.

Since

$$\frac{F^{-1}(\Omega Y) + \Omega U}{\ker F \cap z^{-1}\Omega_\infty U} \simeq Z(F) \oplus \mathcal{W}(\ker F)$$

is finite dimensional and every equivalence class contains a strictly proper function there exists a rational function $H(zI - \Lambda)^{-1}G$ such that for every strictly proper rational function $g \in F^{-1}(\Omega Y) + \Omega U$ there exists a vector α such that

$$g - H(zI - \Lambda)^{-1}G\alpha \in \ker F \cap z^{-1}\Omega_\infty U .$$

(We might obviously assume that (H, Λ) is an observable, while (Λ, G) is a controllable pair.)

3.1.1 The linear space $Z(F) \oplus \mathcal{W}(\ker F)$

The argument above can be amplified to the following theorem.

Theorem 3.3. (i) Assume that the pair (C, A) is observable. Then there exists a pair (H_{fzk}, Λ_{fzk}) and a matrix Π_{fzk} such that for every strictly proper rational q -tuple

$$g \in F^{-1}(\Omega Y) + \Omega U$$

there exists a vector α for which

$$g(z) - H_{fzk}(zI - \Lambda_{fzk})^{-1}\alpha \in \ker F \cap z^{-1}\Omega_\infty U ,$$

furthermore equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{fzk} \\ H_{fzk} \end{bmatrix} = \begin{bmatrix} \Pi_{fzk}\Lambda_{fzk} \\ 0 \end{bmatrix} , \quad (3.15)$$

holds, and the kernel of Π_{fzk} is trivial.

(ii) If (Π_1, H_1, Λ_1) provide a solution of (3.15) then there exists a triple (Π', H', Λ') solving equation (3.15) such that

$$\ker \Pi' = \{0\} \ , \quad \text{Im} \Pi' = \text{Im} \Pi_1$$

and the columns of $H' (zI - \Lambda')^{-1}$ generate the same equivalence classes with respect to $\ker F \cap z^{-1} \Omega_\infty U$ as those of $H_1 (zI - \Lambda_1)^{-1}$.

Moreover, if (C, A) is observable and $\text{Im} \Pi_1 \subset < A \mid B >$ then the inclusion

$$\text{Im} (\Pi_1) \subset \text{Im} (\Pi_{fzk})$$

holds, where Π_{fzk} is determined by part (i).

(iii) Assume that the pair (C, A) is observable. Consider a triplet (Π_1, H_1, Λ_1) providing a solution of (3.15). Let G_1 be a column vector. Then the function $H_1 (zI - \Lambda_1)^{-1} G_1$ is in $F^{-1}(\Omega U) + \Omega U$ if and only if

$$\Pi_1 G_1 \in \langle A \mid B \rangle \ .$$

where $\langle A \mid B \rangle$ denotes the reachability subspace of the state-space.

PROOF. (i) Consider a basis in the finite-dimensional space of equivalence classes

$$(F^{-1}(\Omega Y) + \Omega U) \ / \ (\ker F \cap z^{-1} \Omega_\infty U)$$

and pick up strictly proper rational functions from their equivalence classes. Using these rational q -tuples form a matrix-valued strictly proper rational function with minimal realization

$$\tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{G} \ .$$

Due to the assumption that its columns generate a basis in $Z(F) \oplus \mathcal{W}(\ker F)$ for every strictly proper rational function $g \in F^{-1}(\Omega Y) + \Omega U$ there exists a vector α such that

$$g(z) - \tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{G} \alpha \in (\ker F \cap z^{-1} \Omega_\infty U) \ .$$

Theorem 3.1 (i) – using the observability of the pair (C, A) – implies that there exists a matrix $\tilde{\Pi}$ for which equation (3.6) holds.

The identity $z\tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{G} = \tilde{H}\tilde{G} + \tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{\Lambda}\tilde{G}$ implies that

$$\tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{\Lambda}\tilde{G} \in F^{-1}(\Omega Y) + \Omega U \ .$$

Thus there exists a matrix Λ_{fzk} such that

$$\tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{\Lambda}\tilde{G} - \tilde{H} \left(zI - \tilde{\Lambda} \right)^{-1} \tilde{G}\Lambda_{fzk} \in (\ker G \cap z^{-1} \Omega_\infty U) \ .$$

On the other hand equation (3.5) implies that

$$\begin{aligned} [D + C(zI - A)^{-1}B] & \left[\tilde{H}(zI - \tilde{\Lambda})^{-1}\tilde{\Lambda}\tilde{G} - \tilde{H}(zI - \tilde{\Lambda})^{-1}\tilde{G}\Lambda_{fzk} \right] \\ & = -C(zI - A)^{-1}\tilde{\Pi}(\tilde{\Lambda}\tilde{G} - \tilde{G}\Lambda_{fzk}) . \end{aligned}$$

Invoking again the observability of the pair (C, A) we get that

$$\tilde{\Pi}(\tilde{\Lambda}\tilde{G} - \tilde{G}\Lambda_{fzk}) = 0 .$$

Straightforward calculation gives that

$$[D + C(zI - A)^{-1}B] \left[\tilde{H}(zI - \tilde{\Lambda})^{-1}\tilde{G} - \tilde{H}\tilde{G}(zI - \Lambda_{fzk})^{-1} \right] = 0$$

Set

$$H_{fzk} = \tilde{H}\tilde{G}, \quad \Pi_{fzk} = \tilde{\Pi}\tilde{G} .$$

Then the columns of $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ determine the same equivalence classes as those of $\tilde{H}(zI - \tilde{\Lambda})^{-1}\tilde{G}$, thus they form a basis in $Z(F) \oplus \mathcal{W}(\ker F)$, and equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{fzk} \\ H_{fzk} \end{bmatrix} = \begin{bmatrix} \Pi_{fzk}\Lambda_{fzk} \\ 0 \end{bmatrix}$$

is satisfied. The fact that the columns of $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ form a basis implies that for any non-zero vector α

$$[D + C(zI - A)^{-1}B] H_{fzk}(zI - \Lambda_{fzk})^{-1}\alpha = -C(zI - A)^{-1}\Pi_{fzk}\alpha \neq 0 ,$$

thus $\Pi_{fzk}\alpha \neq 0$. In other words

$$\ker \Pi_{fzk} = \{0\} .$$

(ii) Consider any solution of the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \Pi_1\Lambda_1 \\ 0 \end{bmatrix} . \quad (3.16)$$

Define the matrix Π' in such a way that its column vectors form a basis in $\text{Im}\Pi_1$. Then there are matrices α, β such that

$$\Pi_1\alpha = \Pi', \quad \Pi'\beta = \Pi_1 .$$

Equation $\Pi'\beta\alpha = \Pi'$ and $\ker\Pi' = \{0\}$ imply that $\beta\alpha = I$. Set

$$H' = H_1\alpha, \quad \Lambda' = \beta\Lambda_1\alpha .$$

Multiplying equation (3.16) from the right by α we obtain that the triplet (Π', H', Λ') provides a solution of (3.16), as well.

Defining a matrix α_1 with column vectors forming a basis in $\ker \Pi_1$, we get that the matrix $[\alpha, \alpha_1]$ is regular. Now equation (3.16) implies that

$$F(z)H_1(zI - \Lambda_1)^{-1} = -C(zI - A)^{-1}\Pi_1,$$

thus

$$F(z)H_1(zI - \Lambda_1)^{-1}\alpha_1 = 0,$$

i.e. the columns of $H_1(zI - \Lambda_1)^{-1}\alpha_1$ are in $\ker F \cap z^{-1}\Omega_\infty U$. On the other hand

$$F(z)\left(H'(zI - \Lambda')^{-1} - H_1(zI - \Lambda_1)^{-1}\alpha\right) = -C(zI - A)^{-1}\Pi' + C(zI - A)^{-1}\Pi_1\alpha = 0,$$

proving the first part of (ii).

Now – using the observability of the pair (C, A) – we show that (H', Λ') is observable, as well. In fact, if for some ξ the identities $H'\xi = 0$, $\Lambda'\xi = \lambda\xi$ holds, then equation (3.16) implies that

$$A\Pi'\xi = \Pi'\Lambda'\xi = \lambda\Pi'\xi, \quad C\Pi'\xi = 0.$$

From the observability of (C, A) we obtain that $\Pi'\xi = 0$, implying that $\xi = 0$, proving the observability (H', Λ') .

Since – according to our assumption – $\text{Im } \Pi' = \text{Im } \Pi_1 \subset \langle A \mid B \rangle$ we can apply Theorem 3.2 (ii). From this we obtain that the columns of $H'(zI - \Lambda')^{-1}$ are in $F^{-1}(\Omega Y) + \Omega U$. Consequently, there exists a matrix α' such that

$$H'(zI - \Lambda')^{-1} - H_{fzk}(zI - \Lambda_{fzk})^{-1}\alpha' \in \ker F \cap z^{-1}\Omega_\infty U.$$

As before, from this it follows that $\Pi' = \Pi_{fzk}\alpha'$, in other words

$$\text{Im } \Pi_1 = \text{Im } \Pi' \subset \text{Im } \Pi_{fzk},$$

proving the maximality of $\text{Im } \Pi_{fzk}$.

(iii) If the function $H_1(zI - \Lambda_1)^{-1}G_1$ is in $F^{-1}(\Omega Y) + \Omega U$ then there exists a polynomial g such that

$$F(z)(H_1(zI - \Lambda_1)^{-1}G_1 + g(z)) = q(z)$$

is also a polynomial. Using equation (3.16) we obtain that

$$C(zI - A)(Bg(z) - \Pi_1 G_1) = q(z) - Dg(z).$$

The observability of the pair (C, A) implies that $(zI - A)^{-1}(Bg(z) - \Pi_1 G_1)$ is a polynomial, as well. Denote this by ψ . Rearranging the terms we get that

$$\Pi_1 G_1 = Bg(z) - (zI - A)\psi(z),$$

proving that $\Pi_1 G_1 \in \langle A \mid B \rangle$.

Conversely, if $\Pi_1 G_1 \in \langle A \mid B \rangle$ then there exist two polynomials g, ψ such that $\Pi_1 G_1 = Bg(z) - (zI - A)\psi(z)$. Straightforward calculation gives that

$$(D + C(zI - A)^{-1})(H_1(zI - \Lambda_1)^{-1}G_1 + g(z)) = C\psi(z) + Dg(z),$$

thus $H_1(zI - \Lambda_1)^{-1}G_1 \in F^{-1}(\Omega Y) + \Omega U$, concluding the proof (iii). ■

For later use it is worth summarizing part (i) and (ii) in the following corollary which was proved e.g. partly in Theorem 2 in [1] without the identification of the zero directions but under more general assumptions.

Corollary 3.2. *Assume that the pair (C, A) is observable, and (A, B) is controllable. Then for an observable pair (H, Λ) the columns of $H(zI - \Lambda)^{-1}$ form a basis in $Z(F) \oplus \mathcal{W}(\ker F)$ if and only if $\text{Im}\Pi$ is maximal and $\ker \Pi = \{0\}$, where Π (together with H, Λ) provide a solution of (3.10).*

Let us note that according to Lemma 3.1 the maximality of $\text{Im}(\Pi)$ can be expressed as $\text{Im}(\Pi) = \mathcal{V}^*(\Sigma)$.

More generally, without assuming the controllability of (A, B) .

Corollary 3.3. *Assume that the pair (C, A) is observable. Then for an observable pair (H, Λ) the columns of $H(zI - \Lambda)^{-1}$ form a basis in $Z(F) \oplus \mathcal{W}(\ker F)$ if and only if*

$$\text{Im}\Pi = \mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle \quad \text{and} \quad \ker \Pi = \{0\} ,$$

where Π (together with H, Λ) provide a solution of (3.6).

We get immediately – using the notation introduced above – that

$$\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle = \text{Im}\Pi_{fzk} . \quad (3.17)$$

REMARK 3.1 Let us observe that even the maximal solution triplet (Π, H, Λ) of equation (3.10) is not unique. Although the subspace $\text{Im}(\Pi_{\max}) = \mathcal{V}^*(\Sigma) = \mathcal{V}^*(A, B, C, D)$ is given by the *realization* of the transfer function F , so without loss of generality we might fix a basis in it, determining this way the matrix Π_{\max} , but even for a *fixed* Π the matrices Λ and H are not necessarily uniquely defined. Obviously, if Λ_1, Λ_2 and H_1, H_2 are two solutions (for the same Π) then equation

$$\begin{bmatrix} B \\ D \end{bmatrix} (H_1 - H_2) = \begin{bmatrix} \Pi(\Lambda_1 - \Lambda_2) \\ 0 \end{bmatrix} \quad (3.18)$$

holds. This equation will play an important role in the characterization of $\mathcal{W}(\ker F)$, as we shall see later.

3.1.2 The module $\mathcal{W}(\ker F)$ and the minimal indices of $\ker F$

To characterize the set $\mathcal{W}(\ker F)$ we have to analyze the space $\ker F + \Omega U$. The next theorem provides a description of this set in term of equation (3.6) and the set of *maximal output-nulling reachability subspace* $\mathcal{R}^*(\Sigma)$.

Theorem 3.4. *Assume that (C, A) is observable.*

Let (Λ_1, G_1) be a controllable pair, where G_1 is a column vector.

The function

$$H_1(zI - \Lambda_1)^{-1} G_1 \in (\ker F + \Omega U)$$

in and only if there exists a solution Π_1 of the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \Pi_1 \Lambda_1 \\ 0 \end{bmatrix} \quad (3.19)$$

and

$$\Pi_1 G_1 \in \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma) .$$

Note that the identification of the “kernel indices” to the subspace $\mathcal{R}^*(\Sigma) = \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)$ was already proved in Theorem 5 of [1] for observable systems and extended in Theorem 6 to general systems.

PROOF. If $H_1(zI - \Lambda_1)^{-1}G_1 \in \ker F + \Omega U$ then it is obviously in $F^{-1}(\Omega Y) + \Omega U$ and there exists a polynomial $h_0 + h_1z + \dots + h_jz^j$ such that

$$g = H_1(zI - \Lambda_1)^{-1}G_1 + h_0 + h_1z + \dots + h_jz^j \in \ker F .$$

Applying Theorem 3.2 (i) we get that there exists a matrix Π_1 such that equation (3.19) holds.

Then

$$\begin{aligned} 0 &= [D + C(zI - A)^{-1}B] [H_1(zI - \Lambda_1)^{-1}G_1 + h_0 + h_1z + \dots + h_jz^j] \\ &= D(h_0 + h_1z + \dots + h_jz^j) + C(zI - A)^{-1}[B(h_0 + h_1z + \dots + h_jz^j) - \Pi_1 G_1] \end{aligned}$$

Thus $C(zI - A)^{-1}[B(h_0 + h_1z + \dots + h_jz^j) - \Pi_1 G_1]$ is a polynomial. Using the observability of the pair (C, A) we get that

$$(zI - A)^{-1}[B(h_0 + h_1z + \dots + h_jz^j) - \Pi_1 G_1] = k_0 + k_1z + \dots + k_{j-1}z^{j-1}$$

is also a polynomial.

Writing up the last two equations term by term we obtain the following set of equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} k_0 & k_1 & \dots & k_{j-1} & 0 \\ h_0 & h_1 & \dots & h_{j-1} & h_j \end{bmatrix} = \begin{bmatrix} \Pi_1 G_1 & k_0 & k_1 & \dots & k_{j-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (3.20)$$

In other words, if in the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

starting from the origin the input sequence h_j, h_{j-1}, \dots, h_0 (in this order) is applied then output sequence during these $j+1$ time instants will be zero while the state vector in the $j+2$ step is exactly $\Pi_1 G_1$.

Thus $\Pi_1 G_1$ is in the minimal input-containing set. Equation (3.19) and Lemma 3.1 imply that $\text{Im}(\Pi_1) \subset \mathcal{V}^*(\Sigma)$, consequently $\Pi_1 G_1 \in \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)$.

Conversely, assume that Π_1 (together with H_1, Λ_1) provides a solution of equation (3.19), and $\Pi_1 G_1 \in \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)$. Since $\Pi_1 G_1 \in \mathcal{C}^*(\Sigma)$ there exists a finite sequence denoted by

h_0, h_1, \dots, h_j such that when this is used as an input h_j, h_{j-1}, \dots, h_0 (in this order) then the output is zero while the immediate next state is $\Pi_1 G_1$.

Forming the function

$$g(z) = H_1 (zI - \Lambda_1)^{-1} G_1 + h_0 + h_1 z + \dots + h_j z^j$$

immediate calculation gives that $F(z)g(z) = 0$. (In these calculations equation (3.19) should be used, as well.) Thus the columns of $H_1 (zI - \Lambda_1)^{-1} G_1$ are in $\ker F + \Omega U$, concluding the proof of the theorem. \blacksquare

REMARK 3.2 According to Theorem 3.3 under the assumptions of the observability of the pair (C, A) there exists a pair (H_{fzk}, Λ_{fzk}) such that the columns of $H_{fzk} (zI - \Lambda_{fzk})^{-1}$ generate a basis in $F^{-1}(\Omega Y) + \Omega U$. Now, if Π_{fzk} is given by equation (3.15), then the obvious inclusion $\mathcal{C}^*(\Sigma) \subset \langle A \mid B \rangle$ and Corollary 3.3 imply that

$$\mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma) = \mathcal{C}^*(\Sigma) \cap \text{Im}(\Pi_{fzk}) .$$

According a theorem proven by Wyman and Sain [14] in the space $\mathcal{W}(\ker F)$ the equivalence classes (modulo $\ker F \cap z^{-1}\Omega_\infty U$) of functions

$$\pi_- (z^{-l} v_j), \quad j = 1, \dots, \nu_j, \quad j = 1, \dots, m$$

form a basis, where the q -tuples v_1, \dots, v_m define a minimal polynomial basis in $\ker F$ and $\nu_j = \deg v_j, j = 1, \dots, m$.

Now we are going to characterize these functions in terms of special solutions of equation (3.6).

Theorem 3.5. *Let (C, A) be an observable pair. Assume that the columns of the function $H_{fzk} (zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15).*

Consider now a maximal solution – in terms of α_0 and R_0 – of the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ R_0 \end{bmatrix} = \begin{bmatrix} \Pi_{fzk} \alpha_0 \\ 0 \end{bmatrix} \quad (3.21)$$

(the maximality is meant in the subspace inclusion sense for $\text{Im } R_0$). Then the equivalence classes in $\mathcal{W}(\ker F)$ are determined by the functions

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} \beta$$

where β is any vector in

$$\langle \Lambda_{fzk} \mid \alpha_0 \rangle = \text{Im} \left([\alpha_0, \Lambda_{fzk} \alpha_0, \Lambda_{fzk}^2 \alpha_0, \dots] \right) .$$

REMARK 3.3 Let us note that equation (3.21) in this theorem coincides to the equation (3.18) describing the non-uniqueness of the solutions H, Λ of (3.6) for a fixed matrix Π .

PROOF. First we are going to show that the minimal polynomial basis v_1, \dots, v_m in $\ker F$ generates a solution of (3.21). Set $l_0 = \max_{j=1, \dots, m} \nu_j$. Denote by

$$R(z) = [z^{-\nu_1}v_1, \dots, z^{-\nu_m}v_m] = R_0 + R_1z^{-1} + \dots + R_lz^{-l}.$$

Note that for any rational function $g \in \ker F$ there exists a rational function h such that $g(z) = R(z)h(z)$, and on the other hand $R_0 = V_h$, the highest (column) degree coefficients matrix of the matrix-polynomial $[v_1, \dots, v_m]$. Theorem 2.1 (iv) implies that it is of maximal column rank.

We claim that if for a strictly proper rational q -tuple g there exists a polynomial ψ of degree no greater than r such that

$$g + \psi \in \ker F$$

then ψ can be written as a linear combination of the columns of $\pi_+(z^s R(z))$, $s = 0, 1, \dots, r$

In fact, as we have pointed out the elements $\pi_-(z^{-l}v_j)$, $l = 1, \dots, \nu_j$, $j = 1, \dots, m$ form a basis in $\mathcal{W}(\ker F)$. In terms of the function $R(z)$ this implies that the columns of $\pi_-(z^s R(z))$, $s = 0, 1, \dots, l_0 - 1$ induce a generating system in $\mathcal{W}(\ker F)$. Thus

$$g(z) - \sum_{s=0}^{l_0-1} \pi_-(z^s R(z)c_s) \in \ker F \cap z^{-1}\Omega_\infty U,$$

for some coefficients c_s , $s = 0, \dots, l_0 - 1$. Denote by $h(z) = \sum_{s=0}^{l_0-1} z^s c_s$. Then

$$\pi_-(R(z)h(z)) + \psi(z) \in \ker F.$$

Since the degree of $R(z)$ in z^{-1} is no greater than l_0 , consequently $z^{l_0} [\pi_-(R(z)h(z)) + \psi(z)]$ is a polynomial in $\ker F$.

According to Theorem 2.1 (ii) there exist polynomials $\phi_1, \phi_2, \dots, \phi_m$ such that

$$\sum_{j=1}^m v_j \phi_j = z^{l_0} [\pi_-(R(z)h(z)) + \psi(z)]$$

Now the degree of the right hand side is $l_0 + r$, consequently –using again Theorem 2.1 –

$$\deg \phi_j + \nu_j \leq l_0 + r.$$

Now

$$\begin{aligned} \psi(z) &= \pi_+ \left(z^{-l_0} \sum_{j=1}^m v_j \phi_j \right) = \pi_+ (R(z) [z^{-(l_0-\nu_1)}\phi_1, \dots, z^{-(l_0-\nu_m)}\phi_m]) \\ &= \pi_+ (R(z) \pi_+ [z^{-(l_0-\nu_1)}\phi_1, \dots, z^{-(l_0-\nu_m)}\phi_m]) . \end{aligned}$$

Since $\deg \pi_+ (z^{-(l_0-\nu_j)} \phi_j) \leq r$ we have obtained that ψ can be written as linear combinations of the columns of $\pi_+ (z^s R(z))$, $s = 0, \dots, r$, as claimed.

Now denote by

$$\gamma_0 + \gamma_1 z + \dots + \gamma_r z^r = \pi_+ [z^{-(l_0-\nu_1)} \phi_1, \dots, z^{-(l_0-\nu_m)} \phi_m].$$

Then obviously

$$g(z) - \pi_- \left(\sum_{j=0}^r z^j R(z) \gamma_j \right) \in \ker F, \quad (3.22)$$

as well.

Now for any $r = 0, 1, \dots, l_0 - 1$ the columns of $z^r R(z)$ are in $\ker F$, thus

$$\pi_- (z^r R(z)) \in \ker F + \Omega U \subset F^{-1}(\Omega Y) + \Omega U.$$

Theorem 3.3 (i) implies that there exist a matrix α_r such that

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_r - \pi_- (z^r R(z)) \in \ker F. \quad (3.23)$$

We are going to show that the subspace $\text{Im} ([\alpha_0, \Lambda_{fzk} \alpha_0, \dots, \Lambda_{fzk}^r \alpha_0])$ contains the column-vectors of α_r . Adding the function $z^r R(z)$ we get that

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_r + R_0 z^r + R_1 z^{r-1} + \dots + R_r \in \ker F. \quad (3.24)$$

On the other hand

$$\begin{aligned} & z^r \pi_- (R(z)) - z^r H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_0 \\ &= z^r R(z) - R_0 z^r - (H_{fzk} (zI - \Lambda_{fzk})^{-1} \Lambda_{fzk}^r \alpha_0 + H_{fzk} \Lambda_{fzk}^{r-1} \alpha_0 + \dots + H_{fzk} \alpha_0 z^{r-1}) \in \ker F \end{aligned}$$

Taking the difference

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} (\alpha_r - \Lambda_{fzk}^r \alpha_0) + (R_r - H_{fzk} \Lambda_{fzk}^{r-1} \alpha_0) + \dots + (R_1 - H_{fzk} \alpha_0) z^{r-1} \in \ker F.$$

In other words by adding a polynomial of degree no greater than $r - 1$ to the strictly proper rational function $H_{fzk} (zI - \Lambda_{fzk})^{-1} (\alpha_r - \Lambda_{fzk}^r \alpha_0)$ a function in $\ker F$ is obtained. Consequently, according to the previous argument for some vectors c_0, c_1, \dots, c_{r-1}

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} (\alpha_r - \Lambda_{fzk}^r \alpha_0) - \pi_- \left(\sum_{j=0}^{r-1} z^j R(z) c_j \right) \in \ker F.$$

Equation (3.23) implies that

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} (\alpha_r - \Lambda_{fzk}^r \alpha_0) - H_{fzk} (zI - \Lambda_{fzk})^{-1} \sum_{j=0}^{r-1} \alpha_j c_j \in \ker F.$$

Due to the fact that the columns of $H_{fzk} (zI - \Lambda_{fzk})^{-1}$ generate a basis in the equivalence classes defined modulo $\ker F \cap z^{-1}\Omega_\infty U$ we get that

$$\alpha_r - \Lambda_{fzk}^r \alpha_0 = \sum_{j=0}^{r-1} \alpha_j c_j .$$

Applying this recursively the inclusion

$$\text{Im}(\alpha_r) \subset \text{Im}([\alpha_0, \Lambda_{fzk} \alpha_0, \dots, \Lambda_{fzk}^r \alpha_0])$$

can be derived.

Let us remark that the following converse statement obviously holds. If for some vector β the identity

$$\beta = \sum_{j=0}^r \Lambda_{fzk}^j \alpha_0 c_j$$

holds, then

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} \beta - \pi_- \left(\sum_{j=0}^r z^j R(z) c_j \right) \in \ker F .$$

It remains to characterize the matrix α_0 . Let us recall that the columns of the proper rational function $H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_0 + R_0$ are in $\ker F$. On the other hand – using equation (3.6)

$$\begin{aligned} (D + C(zI - A)^{-1} B) (H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_0 + R_0) \\ = DR_0 + C(zI - A)^{-1} [BR_0 - \Pi_{fzk} \alpha_0] , \end{aligned}$$

implying that

$$\begin{aligned} DR_0 &= 0 \\ BR_0 - \Pi_{fzk} \alpha_0 &= 0 \end{aligned}$$

proving that (3.21) holds for the matrices R_0, α_0 where $R_0 = V_h$ and α_0 is obtained as the solution of $H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_0 - \pi_- (R(z)) \in \ker(F)$ (or as $R_0 + H_{fzk} (zI - \Lambda_{fzk})^{-1} \alpha_0 \in \ker(F)$).

To prove the maximality of $\text{Im} R_0$ observe that, if for some vectors β and γ_0 the equations

$$D\gamma_0 = 0 , \quad B\gamma_0 - \Pi_{fzk} \beta = 0$$

hold then the rational function $H_{fzk} (zI - \Lambda_{fzk})^{-1} \beta + \gamma_0$ is in $\ker F$, thus the previous argument applied for $r = 0$ and $g(z) = H_{fzk} (zI - \Lambda_{fzk})^{-1} \beta$ gives that

$$H_{fzk} (zI - \Lambda_{fzk})^{-1} \beta - \pi_- (R(z) c_0) \in \ker F$$

holds, implying that

$$\beta = \alpha_0 c_0 .$$

(The case $\beta = 0$ corresponds to the situation when the constant vector γ_0 is in the kernel of F . Note that $\pi_-(\gamma_0) = 0$, consequently in this case the corresponding equivalence class in $\mathcal{W}(\ker F)$ is zero.)

Thus – fixing Π_{fzk} – a maximal solution of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ R_0 \end{bmatrix} = \begin{bmatrix} \Pi_{fzk}\alpha_0 \\ 0 \end{bmatrix}$$

should be considered (the maximality is meant in the subspace inclusion sense for $\text{Im}(R_0)$) and for any vector β in

$$\text{Im}([\alpha_0, \Lambda_{fzk}\alpha_0, \Lambda_{fzk}^2\alpha_0, \dots])$$

the strictly proper rational function

$$H_{fzk}(zI - \Lambda_{fzk})^{-1}\beta$$

generates an equivalence class in $\mathcal{W}(\ker F)$. ■

REMARK 3.4 Note that an immediate consequence of the previous theorem that the minimal indices ν_1, \dots, ν_m of the minimal polynomial basis in $\ker F$ coincide with the controllability indices of the pair $(\Lambda_{fzk}, \alpha_0)$.

This result should be considered in parallel to Theorem 5 in [1] where the minimal indices corresponding to a minimal polynomial basis in $\ker F$ are also identified with the controllability indices of a pair of suitably chosen matrices. There these matrices are obtained using a feedback transformation. In addition to these Corollary 3 of the same paper shows that these minimal indices corresponding to the $\ker F$ are invariant under feedback transformation and output injection, as well. (I.e. for the systems (A, B, C, D) , $(A + BL, B, C + DL, D)$ and $(A + LC, B + LD, C, D)$ these minimal indices coincide.

Corollary 3.4. *Let (C, A) be an observable pair. Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15).*

Then

$$\mathcal{C}^*(\Sigma) \cap \text{Im}(\Pi_{fzk}) = \Pi_{fzk} \langle \Lambda_{fzk} \mid \alpha_0 \rangle .$$

It is worth pointing out that the following statement which was already present in [2] is also an immediate corollary of the previous theorem.

Corollary 3.5. *Let (C, A) be an observable pair. Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15).*

Then the subspace $\mathcal{W}(\ker F)$ is trivial if and only if

$$\text{Im}(\Pi_{fzk}) \cap \{B\eta \mid D\eta = 0\} = \{0\}.$$

PROOF. This is immediate from the previous theorem giving that $\mathcal{W}(\ker F) = \{0\}$ if and only if the only solution of (3.21) is $R_0 = 0, \alpha_0 = 0$. ■

In some cases the following form of this corollary can be also of use which follows immediately from the inclusion $\text{Im}B \subset \langle A \mid B \rangle$ and the identity $\text{Im}\Pi_{fzk} = \text{Im}\Pi_{max} \cap \langle A \mid B \rangle$.

Corollary 3.6. *Assume that (C, A) is an observable pair. Consider a maximal solution $(\Pi_{max}, H_{max}, \Lambda_{max})$ of equation (3.10).*

Then the subspace $\mathcal{W}(\ker F)$ is trivial if and only if

$$\mathcal{V}^*(\Sigma) \cap \{B\eta \mid D\eta = 0\} = \text{Im}(\Pi_{max}) \cap \{B\eta \mid D\eta = 0\} = \{0\}.$$

REMARK 3.5 Let us recall (see e.g. [1]) that the function F is left-invertible if and only if both $\mathcal{W}(\ker F)$ and $\ker \begin{bmatrix} B \\ D \end{bmatrix}$ are trivial.

The following corollary is also immediate from the previous theorem.

Corollary 3.7. *Assume that (C, A) is observable, (A, B) is controllable. Then the subspace $Z(F)$ is trivial if and only if the pair $(\Lambda_{fzk}, \alpha_0)$ from equations (3.15), (3.21) is controllable.*

REMARK 3.6 Let us emphasize that according to Corollary 3.3 any pair (H, Λ) can be used as a starting point in Theorem 3.5 for which the corresponding solution Π of (3.6) satisfies that $\ker \Pi = \{0\}$ and $\text{Im}\Pi = \mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle$. For a fixed Π with these properties the nonuniqueness of maximal solution of (3.21) is determined by a nonsingular matrix multiplier from the right. Thus for a fixed Π (with $\ker \Pi = \{0\}$, $\text{Im}\Pi = \mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle$) all solution of (3.15) and (3.21) can be described as $(H + R_0\beta, \Lambda + \alpha_0\beta)$, and $(R_0\gamma, \alpha_0\gamma)$ where β is an arbitrary matrix, γ is an arbitrary nonsingular matrix, (H, Λ) and R_0, α_0 are particular solutions of these equations.

REMARK 3.7 Furthermore, the identity $\text{Im}\Pi_{fzk} = \text{Im}\Pi_{max} \cap \langle A \mid B \rangle$ implies that R_0 and α_0 can be determined starting with the matrix Π_{max} instead of Π_{fzk} . In fact, consider a maximal solution $(\Pi_{max}, H_{max}, \Lambda_{max})$ of (3.6) with $\ker(\Pi_{max}) = \{0\}$ and using Π_{max} consider a maximal solution $\widetilde{R}_0, \widetilde{\alpha}_0$ of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ \widetilde{R}_0 \end{bmatrix} = \begin{bmatrix} \Pi_{max} \widetilde{\alpha}_0 \\ 0 \end{bmatrix} \quad (3.25)$$

assuming that $\ker(\widetilde{R}_0) = \{0\}$, where the maximality is meant in the subspace inclusion sense for $\text{Im}(\Pi_{max})$ and $\text{Im}(\widetilde{R}_0)$.

Multiplying Π_{max} from the right by a nonsingular matrix and $\widetilde{\alpha}_0$ from the right by its inverse we might assume that Π_{max} has the following form $\Pi_{max} = [\Pi_{fzk}, \Pi']$. Partition $\widetilde{\alpha}_0$ accordingly: $\widetilde{\alpha}_0 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ Now

$$B\widetilde{R}_0 = \Pi_{fzk}\alpha_1 + \Pi'\alpha_2.$$

The obvious inclusion $\text{Im}B \subset \langle A \mid B \rangle$ gives that the columns of the matrix above should be in $\langle A \mid B \rangle \cap \text{Im}(\Pi_{max}) = \text{Im}(\Pi_{fzk})$. Thus $\Pi'\alpha_2 = 0$. i.e. $\alpha_2 = 0$. The equations

$B\widetilde{R}_0 = \Pi_{fzk}\alpha_1$, $D\widetilde{R}_0 = 0$ and the maximality of the solution $\widetilde{R}_0, \widetilde{\alpha}_0$ implies that after a multiplication from the right by a nonsingular matrix we can achieve that

$$\widetilde{R}_0 = R_0, \quad \alpha_1 = \alpha_0.$$

Moreover, applying the same nonsingular matrix multiplication from the right to the equation (3.10) we might again assume that $\Pi_{max} = [\Pi_{fzk}, \Pi']$. Partition H_{max} and Λ_{max} accordingly.

$$H_{max} = [H_1, H_2], \quad \Lambda_{max} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

We obtain that

$$A\Pi_{fzk} + BH_1 = \Pi_{fzk}\Lambda_{11} + \Pi'\Lambda_{21}.$$

But equation (3.15) shows that the columns of $A\Pi_{fzk}$ are in the subspace generated by $\text{Im}B$ and $\text{Im}\Pi_{fzk}$. Similar argument as before gives that $\Pi'\Lambda_{21} = 0$, i.e. $\Lambda_{21} = 0$. It follows that H_1, Λ_{11} provide a solution of (3.15), consequently they can be denoted by H_{fzk}, Λ_{fzk} .

Let us observe that

$$\begin{aligned} R_0 + H_{max}(zI - \Lambda_{max})^{-1}\widetilde{\alpha}_0 = \\ R_0 + [H_{fzk}, H_2] \left(z \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Lambda_{fzk} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix} = \\ R_0 + H_{fzk}(zI - \Lambda_{fzk})^{-1}\alpha_0. \end{aligned} \quad (3.26)$$

REMARK 3.8 Let us introduce the notation:

$$K_0(z) = R_0 + H_{fzk}(zI - \Lambda_{fzk})^{-1}\alpha_0.$$

As we have already pointed out the columns of this function are in the kernel of F . Moreover, the columns of $\pi_-(z^r K_0(z))$, $r \geq 0$ generate $\mathcal{W}(\ker F)$.

Note that the realization above of K_0 is – in general – non-minimal. Although the observability of the pair (C, A) implies that (H_{fzk}, Λ_{fzk}) is observable, as well, the controllability of $(\Lambda_{fzk}, \alpha_0)$ in general does not hold.

Let us emphasize that the function K_0 is defined via fixing a particular solution of (3.15) and also of (3.21). As we have pointed out in Remark 3.6 all solution can be obtained from these. Let us observe that – using the notations from Remark 3.6

$$[R_0\gamma + (H_{fzk} + R_0\beta)(zI - (\Lambda_{fzk} + \alpha_0\beta))^{-1}\alpha_0\gamma] \gamma^{-1} (I - \beta(zI - \Lambda_{fzk})^{-1}\alpha_0) = K_0(z),$$

where the function $\gamma^{-1} (I - \beta(zI - \Lambda_{fzk})^{-1}\alpha_0)$ is proper with proper inverse.

The following proposition explicitly shows how the columns of the function K_0 generate the kernel of F .

Proposition 3.3. *Assume that for the proper rational q -tuple g the identity*

$$F(z)g(z) = 0$$

holds. Then there exists a proper rational function $h(z)$ such that

$$g(z) = K_0(z)h(z) ,$$

i.e. the columns of K_0 generate the kernel of F over the field of rational functions.

PROOF. Assume that the realization of F given by

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

where (C, A) is an observable pair, furthermore the realization of g is given by

$$g(z) \sim \left(\begin{array}{c|c} \lambda & \beta \\ \hline \gamma & \delta \end{array} \right) ,$$

where (λ, β) is a controllable pair.

Evaluating equation $Fg = 0$ at infinity we obtain that $D\delta = 0$. Now, the equation can be written in the following form

$$(D + C(zI - A)^{-1}B) \gamma (zI - \lambda)^{-1} \beta = -C(zI - A)^{-1} B \delta .$$

According to Proposition 3.1 there exists a solution ρ of the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \rho \\ \gamma \end{bmatrix} = \begin{bmatrix} \rho\lambda \\ 0 \end{bmatrix}$$

for which $\rho\beta = B\delta$ holds. This latter together with $D\delta = 0$ can be written in the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ \delta \end{bmatrix} = \begin{bmatrix} \rho\beta \\ 0 \end{bmatrix} .$$

Using the maximality of Π_{\max} and R_0 we get that

$$\rho = \Pi_{\max} \xi , \quad \text{and consequently} \quad \delta = R_0 \eta$$

for some matrix ξ and some vector η . Substituting into $\rho\beta = B\delta$ we obtain that $\Pi_{\max} \xi \beta = BR_0 \eta = \Pi_{\max} \tilde{\alpha}_0 \eta$ implying that

$$\xi \beta = \tilde{\alpha}_0 \eta .$$

Multiplying the equation (3.10) from the right by ξ and (3.25) from the right by η and taking the differences with the previous equations we arrive at the following equations:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ \gamma - H_{\max} \xi \end{bmatrix} \begin{bmatrix} \Pi_{\max} (\xi \lambda - \Lambda_{\max} \xi) \\ 0 \end{bmatrix}$$

implying that there exists a matrix ζ for which

$$\gamma - H_{\max}\xi = R_0\zeta \quad (3.27)$$

$$\xi\lambda - \Lambda_{\max}\xi = \tilde{\alpha}_0\zeta \quad (3.28)$$

hold.

Now, define

$$h(z) = \eta + \zeta(zI - \lambda)^{-1}\beta.$$

Then straightforward calculation – using the identity (3.26) – gives that

$$K_0(z)h(z) = g(z),$$

concluding the proof of the proposition. ■

REMARK 3.9 Let us return to the non-uniqueness of the solution of equation (3.15). Assume that the pair (C, A) is observable. Consider a solution $(\Pi_{fzk}, H_{fzk}, \Lambda_{fzk})$ of (3.15) for which $\text{Im}\Pi_{fzk} = \mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle$, $\ker \Pi_{fzk} = \{0\}$ and a maximal solution (α_0, R_0) of (3.21). The maximality of this solution implies that – fixing the matrix Π_{fzk} – any solution of (3.15) has the form

$$(\Pi_{fzk}, H_{fzk} + R_0\beta, \Lambda_{fzk} + \alpha_0\beta),$$

where β is an arbitrary matrix (of appropriate size).

Without loss of generality we might assume that the matrices Λ_{fzk} and α_0 are of the form

$$\Lambda_{fzk} = \begin{bmatrix} \Lambda_k & \Lambda_{kf} \\ 0 & \Lambda_f \end{bmatrix}, \quad \alpha_0 = \begin{bmatrix} \alpha_k \\ 0 \end{bmatrix}, \quad (3.29)$$

where the pair (Λ_k, α_k) is controllable. Accordingly,

$$\Pi_{fzk} = [\Pi_k, \Pi_f], \quad H_{fzk} = [H_k, H_f], \quad \beta = [\beta_1, \beta_2]. \quad (3.30)$$

(Note, that this transformation does not affect the choice of R_0 .)

Observe that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_k \\ H_k \end{bmatrix} = \begin{bmatrix} \Pi_k \Lambda_k \\ 0 \end{bmatrix}, \quad (3.31)$$

and

$$\begin{bmatrix} B \\ D \end{bmatrix} R_0 = \begin{bmatrix} \Pi_k \alpha_k \\ 0 \end{bmatrix}. \quad (3.32)$$

For later use it is worth pointing out that the controllability of the pair (Λ_k, α_k) and Corollary 3.4 imply that the identity

$$\text{Im}(\Pi_k) = \mathcal{C}^*(\Sigma) \cap \text{Im}(\Pi_{fzk}) = \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma) = \mathcal{R}^*(\Sigma) \quad (3.33)$$

holds.

Notice also that

$$\Lambda_{fzk} + \alpha_0\beta = \begin{bmatrix} \Lambda_k + \alpha_k\beta_k & \Lambda_{kf} + \alpha_k\beta_f \\ 0 & \Lambda_f \end{bmatrix}$$

and

$$H_{fzk} + R_0\beta = [H_k + R_0\beta_k, H_f + R_0\beta_f] .$$

Consequently, invoking Theorem 3.5 we get that the columns of the function

$$(H_k + R_0\beta_k)(zI - (\Lambda_k + \alpha_k\beta_k))^{-1}$$

generate a basis in $\mathcal{W}(\ker F)$, while (factoring out $\mathcal{W}(\ker F)$) the columns of

$$(H_f + R_0\beta_f)(zI - \Lambda_f)^{-1}$$

generate a basis in $Z(F)$.

This latter observation justifies the following definition

Definition 3.1. *Under the assumptions of the previous remark the matrix Λ_f is called finite zero matrix of the function F . Its eigenvalues are the so-called finite (or transmission) zeros of F .*

The eigenvalues of Λ_k (or of $(\Lambda_k + \alpha_k\beta_k)$) are called virtual zeros of F .

The expression *virtual zero* refers to the fact that choosing the matrix β in an appropriate way the eigenvalues of $\Lambda_k + \alpha_k\beta_k$ can be moved around in the complex plane.

REMARK 3.10 Let us introduce the notation

$$K_\beta(z) = R_0 + (H_{fzk} + R_0\beta)(zI - (\Lambda_{fzk} + \alpha_0\beta))^{-1} \alpha_0 .$$

This function has obviously the same properties as K_0 , namely, its columns are in the kernel of F , and the columns of $\pi_-(z^r K_\beta(z))$, $r \geq 0$ generate $\mathcal{W}(\ker F)$.

3.1.3 Choosing K_β as a tall inner-function

The following theorem shows that the rational function $K_0(z) = R_0 + H_{fzk}(zI - \Lambda_{fzk})^{-1} \alpha_0$ being equivalent to $R(z)$ and playing an important role in the proof of Theorem 3.5 can be chosen to be a tall inner function. To this aim we are going to use the property that for a fixed matrix Π all solutions of (3.15) can be given in the form $\Lambda + \alpha_0\beta$, $H + R_0\beta$, where β is arbitrary.

Theorem 3.6. *Let (C, A) be an observable pair. Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15). Consider a maximal solution – in terms of α_0 and R_0 – of the equation (3.21) assuming – w.l.o.g. – that the column-vectors of the matrix R_0 are orthonormal. Then there exists a matrix β such that the function*

$$K_\beta(z) = R_0 + (H_{fzk} + R_0\beta)(zI - (\Lambda_{fzk} + \alpha_0\beta))^{-1} \alpha_0$$

is a tall inner (in continuous time sense) function.

PROOF. We might assume that the matrices are partitioned according to (3.29) and (3.30) Then

$$K_\beta(z) = R_0 + (H_k + R_0\beta_k)(zI - (\Lambda_k + \alpha_k\beta_k))^{-1}\alpha_k.$$

(Especially, the value of β_f has no effect on the function K_β .)

Obviously the equations

$$\sigma(\Lambda_k + \alpha_k\beta_k) + (\Lambda_k + \alpha_k\beta_k)^*\sigma + (H_k + R_0\beta_k)^*(H_k + R_0\beta_k) = 0 \quad (3.34)$$

$$\alpha_k^*\sigma + R_0^*(H_k + R_0\beta_k) = 0 \quad (3.35)$$

$$R_0^*R_0 = I \quad (3.36)$$

imply the equation $K_\beta^*(z)K_\beta(z) = I$.

Due to the fact that the columns of R_0 are orthonormal the third equation trivially holds.

The second equation gives that

$$\beta_k = -\alpha_k^*\sigma - R_0^*H_k. \quad (3.37)$$

Substituting this expression into the first equation the following Riccati-equation

$$\sigma(\Lambda_k - \alpha_k R_0^* H_k) + (\Lambda_k - \alpha_k R_0^* H_k)^* \sigma - \sigma \alpha_k \alpha_k^* \sigma + H_k^* (I - R_0 R_0^*) H_k = 0 \quad (3.38)$$

is obtained.

The controllability of the pair (Λ_k, α_k) implies that equation (3.38) has a unique positive semidefinite solution.

Next we prove that any solution σ of this equation is invertible. Obviously, if $\xi \in \ker \sigma$ then – multiplying by ξ^* from the left and by ξ from the right the equation

$$(I - R_0 R_0^*) H_k \xi = 0$$

is obtained. Now multiplying only from the right by ξ we get that $\ker \sigma$ is $(\Lambda_k - \alpha_k R_0^* H_k)$ -invariant. Choosing ξ to be an eigenvector of this matrix

$$(\Lambda_k - \alpha_k R_0^* H_k) \xi = \lambda \xi,$$

and using (3.31) and (3.32) we obtain that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_k \xi \\ (H_k - R_0 R_0^* H_k) \xi \end{bmatrix} = \begin{bmatrix} \Pi_k (\Lambda_k - \alpha_k R_0^* H_k) \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \Pi_k \xi \\ 0 \end{bmatrix}.$$

Invoking the observability of the pair (C, A) we get that $\Pi_k \xi = 0$. But according to our assumption the column vectors of Π_k are linearly independent, thus $\xi = 0$, proving the invertibility of σ .

It remains to prove the analyticity of K_β on the right half plane. If ξ is an eigenvector of $\Lambda_k + \alpha_k \beta_k$ i.e.

$$(\Lambda_k + \alpha_k \beta_k) \xi = \lambda \xi,$$

then

$$2\operatorname{Re} \lambda \xi^* \sigma \xi + \xi^* \sigma \alpha_k \alpha_k^* \sigma \xi + \xi^* H_k^* (I - R_0 R_0^*) H_k \xi = 0.$$

Thus $\operatorname{Re} \lambda \leq 0$. If $\operatorname{Re} \lambda = 0$, then $(I - R_0 R_0^*) H_k \xi = 0$ and $\alpha_k^* \sigma \xi = 0$. Using again equations (3.31) and (3.32) we obtain that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_k \xi \\ H_k \xi - R_0 (\alpha_k^* \sigma + R_0^* H_k) \xi \end{bmatrix} = \begin{bmatrix} \Pi_k (\Lambda_k - \alpha_k (\alpha_k^* \sigma + R_0^* H_k)) \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \Pi_k \xi \\ 0 \end{bmatrix}.$$

Invoking the observability of the pair (C, A) and $\ker \Pi_k = \{0\}$ we get – similarly as before – $\xi = 0$. Thus the matrix $\Lambda_k + \alpha_k \beta_k$ is asymptotically stable (in continuous time sense), proving that the function K_β is inner. \blacksquare

The inner function K_β will be called as “right-kernel” inner function.

3.1.4 Zero structure of a tall inner function

As an immediate application of Theorems 3.3 and 3.5 let us consider the zero structure of a tall inner function. In order to emphasize that in this subsection a special case is considered let us denote this tall inner function by $Q(z) = D_Q + C_Q (zI - A_Q)^{-1} B_Q$. Assume that the pair (C_Q, A_Q) is observable and all the eigenvalues of the matrix A_Q have negative real part. Consider a square inner extension of Q in the form $[Q, \tilde{Q}]$ assuming that a realization of \tilde{Q} is given as $\tilde{Q}(z) = \tilde{D}_Q + C_Q (zI - A_Q)^{-1} \tilde{B}_Q$.

As it is well-known this extension can be obtained in the following way. Consider the solution P of the Lyapunov equation

$$PA_Q + A_Q^* P + C_Q^* C_Q = 0. \quad (3.39)$$

P is uniquely determined and positive definite. The matrix \tilde{D}_Q provides a unitary extension of D_Q , i.e. $\begin{bmatrix} D_Q & \tilde{D}_Q \end{bmatrix}$ is a unitary matrix. (In other words, the orthonormal vectors formed by the columns of D_Q are extended to an orthonormal basis.) Then

$$\tilde{B}_Q = -P^{-1} C_Q^* \tilde{D}_Q.$$

Note that $B_Q = -P^{-1} C_Q^* D_Q$. In other words the identity

$$\begin{bmatrix} B_Q & \tilde{B}_Q \end{bmatrix} \begin{bmatrix} D_Q \\ \tilde{D}_Q \end{bmatrix}^* + P^{-1} C_Q^* = 0$$

holds.

Proposition 3.4. *Let Q be an tall inner function (in continuous time sense) with the realization above. Assume that the pair (C_Q, A_Q) is observable, and all the eigenvalues of A_Q have negative real part.*

Consider a square inner extension $[Q, \tilde{Q}]$ of Q with the realization above. Then

(i) the maximal solution of (3.10) is given by the triplet (Π, H, Λ) where $P\text{Im}(\Pi)$ is the orthogonal complement of the reachability subspace $\langle A_Q | \tilde{B}_Q \rangle$.

$$H = -D_Q^* C_Q \Pi$$

and Λ is determined by the equation

$$(A_Q - B_Q D_Q^* C_Q) \Pi = \Pi \Lambda .$$

The matrix Λ is the finite zero matrix of Q .

(ii) The module $\mathcal{W}(\ker Q)$ is trivial.

(iii) On the subspace $\text{Im}(\Pi)$ the matrices $-P^{-1}A_Q^*P$ and $(A_Q - B_Q D_Q^* C_Q)$ coincide.

PROOF. According the Theorem 3.3 first a maximal solution of equation

$$\begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix} \begin{bmatrix} \Pi \\ H \end{bmatrix} = \begin{bmatrix} \Pi \Lambda \\ 0 \end{bmatrix} \quad (3.40)$$

should be considered.

Multiplying the second equation from the left by D_Q^* we get that $H = -D_Q^* C_Q \Pi$. Substituting this values into the first and the second equation we arrive at the following equations:

$$\begin{aligned} (A_Q - B_Q D_Q^* C_Q) \Pi &= \Pi \Lambda , \\ (I - D_Q D_Q^*) C_Q \Pi &= 0 . \end{aligned}$$

Now $B_Q D_Q^* C_Q = -P^{-1}C_Q^* D_Q D_Q^* C_Q$, consequently the Lyapunov-equation (3.39) above can be written as

$$P (A_Q - B_Q D_Q^* C_Q) + A_Q^* P + C_Q^* (I - D_Q D_Q^*) C_Q = 0 .$$

Multiplying from the right by Π and from the left by P^{-1} we obtain that

$$P^{-1} A_Q^* P \Pi = - (A_Q - B_Q D_Q^* C_Q) \Pi = -\Pi \Lambda .$$

Thus the subspace $\text{Im}(\Pi)$ should be $(A_Q - B_Q D_Q^* C_Q)$ -invariant and on it the matrices $-P^{-1}A_Q^*P$ and $(A_Q - B_Q D_Q^* C_Q)$ coincide. (This proves (iii).)

The identities

$$\begin{aligned} (I - D_Q D_Q^*) C_Q &= \tilde{D}_Q \tilde{D}_Q^* C_Q = -\tilde{D}_Q \tilde{B}_Q^* P \\ A_Q^* P \Pi &= -P \Pi \Lambda \end{aligned}$$

imply that the subspace $P\text{Im}(\Pi)$ should be orthogonal to the reachability subspace $\langle A_Q | \tilde{B}_Q \rangle$.

Conversely, consider orthogonal complement of $\langle A_Q | \tilde{B}_Q \rangle$ and choose the matrix Π is such a way that the columns of $P\Pi$ span the this subspace. In this case then

$$\begin{aligned} \Pi^* P \tilde{B}_Q &= 0 \\ \Pi^* P A_Q &= -\Lambda \Pi^* P \end{aligned}$$

for some matrix Λ . Using the equations above we get that

$$(I - D_Q D_Q^*) C_Q \Pi = 0 ,$$

and from the Lyapunov-equation (3.39) we obtain that

$$(A_Q - B_Q D_Q^* C_Q) \Pi = \Pi \Lambda$$

Thus defining $H = -D_Q^* C_Q \Pi$ we get that (Π, H, Λ) provide a solution of (3.40), proving the first part of (i).

To identify the corresponding Λ as the finite zero matrix – using Corollary 3.3 and Remark 3.9 – we have to prove that $\langle A \mid B \rangle \supset \text{Im}(\Pi)$ and $\mathcal{W}(\ker Q)$ is trivial.

To this aim first consider solutions of the Lyapunov-equations

$$\begin{aligned} A_Q P_1 + P_1 A_Q^* + B_Q B_Q^* &= 0 , \\ A_Q P_2 + P_2 A_Q^* + \tilde{B}_Q \tilde{B}_Q^* &= 0 . \end{aligned}$$

Then invoking that $[Q, \tilde{Q}]$ is a square inner function we get that

$$P_1 + P_2 = P^{-1} . \quad (3.41)$$

The kernel of P_2 determines the orthogonal complement of the reachability subspace $\langle A \mid \tilde{B} \rangle$, while the image of P_1 gives $\langle A \mid B \rangle$. Now –as we have seen – for the maximal solution of (3.40) the identity

$$\text{Im}(P\Pi) = \ker(P_2)$$

holds. In other words

$$\text{Im}(\Pi) = P^{-1} \ker(P_2) .$$

But the equation (3.41) implies that if $\xi \in \ker(P_2)$ then $\xi = P P_1 \xi$, thus

$$\text{Im}(\Pi) = P^{-1} \ker(P_2) \subset \text{Im}(P_1) = \langle A \mid B \rangle .$$

Now consider a maximal solution of (3.40) and solve the equation

$$\begin{bmatrix} B_Q \\ D_Q \end{bmatrix} R_0 = \begin{bmatrix} \Pi \alpha \\ 0 \end{bmatrix} .$$

But the identity $D_Q^* D_Q = I$ implies that $R_0 = 0$, i.e. according to Theorem 3.5 the module $\mathcal{W}(\ker Q)$ is trivial, proving (iii) and finishing the proof of (i), thus concluding the proof of the proposition. \blacksquare

REMARK 3.11 Let us point out two special cases of the proposition above.

- (i) The finite zero module $Z(Q)$ is trivial, if the pair (A_Q, \tilde{B}_Q) is controllable,
- (ii) The finite zero matrix of Q is given by $A_Q - B_Q D_Q^* C_Q$, if Q is a square inner function.

3.1.5 Eliminating $\mathcal{W}(\ker F)$ via factorization

The “right-kernel” inner function K_β constructed in Theorem 3.6 is a tall inner function. Consider its square inner extension. Straightforward computation gives that the function

$$\begin{aligned} K_{\beta, ext} &= [K_\beta, L_\beta] \\ &= [R_0, L_0] + (H_k + R_0\beta_k)(zI - (\Lambda_k + \alpha_k\beta_k))^{-1} [\alpha_k, -\sigma^{-1}(H_k + R_0\beta_k)^* L_0] \end{aligned} \quad (3.42)$$

– where the matrix L_0 is chosen in such a way that the matrix $[R_0, L_0]$ be unitary, and σ is the positive definite solution of the Riccati-equation (3.38) – is a square inner function.

REMARK 3.12 Let us observe that Proposition 3.4 implies (using that the pair (Λ_k, α_k) is reachable) that the finite zero module $Z(L_\beta)$ and the kernel module $\mathcal{W}(\ker L_\beta)$ of L_β is trivial.

Now define the function $F_{\mathbf{r}}$ as follows.

$$F_{\mathbf{r}} = FL_\beta. \quad (3.43)$$

Then $F_{\mathbf{r}}L_\beta^* = FL_\beta L_\beta^* = F(K_\beta K_\beta^* + L_\beta L_\beta^*) = F$, using that $FK_\beta = 0$ and $K_\beta K_\beta^* + L_\beta L_\beta^* = I$.

The following theorem essentially shows that $Z(F_{\mathbf{r}}) = Z(F) \oplus \mathcal{W}(\ker F)$ (they are isomorphic as vector spaces).

Theorem 3.7. *Let (C, A) be an observable pair. Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15). Consider a maximal solution – in terms of α_0 and R_0 – of the equation (3.21) assuming – w.l.o.g. – that the column-vectors of the matrix R_0 are orthonormal and the matrices are partitioned according to (3.29) and (3.30).*

Consider the function $F_{\mathbf{r}}$ defined in (3.43). Then

(i) $F_{\mathbf{r}}$ has the following (in general non-minimal) realization

$$F_{\mathbf{r}} \sim \Sigma_{\mathbf{r}} = \left(\begin{array}{c|c} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ \hline C & DL_0 \end{array} \right) \quad (3.44)$$

where σ is the positive definite solution of the Riccati-equation (3.38).

(ii) denoting by $\mathcal{V}^*(\Sigma_{\mathbf{r}})$, $\mathcal{C}^*(\Sigma_{\mathbf{r}})$ the maximal output-nulling controlled invariant subspace and the minimal input-containing subspace, respectively, of the **realization** of $F_{\mathbf{r}}$ provided in (3.44) we get that

$$\mathcal{V}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}^*(\Sigma) \quad (3.45)$$

$$\mathcal{V}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma_{\mathbf{r}}) = \{0\} \quad (3.46)$$

$$(\mathcal{V}^*(\Sigma) \cap \mathcal{C}^*(\Sigma)) \vee \mathcal{C}^*(\Sigma_{\mathbf{r}}) = \mathcal{C}^*(\Sigma) \quad (3.47)$$

and

$$\mathcal{W}(\ker F_{\mathbf{r}}) = \{0\}, \quad (3.48)$$

(iii) the reachability subspace of the given realization of F contains that of (3.44), i.e.

$$\langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle \subset \langle A \mid B \rangle$$

and if

a) all the eigenvalues of the matrix A have non-positive real part, or

b) the matrices

$$A \quad \text{and} \quad -A^* \quad \text{have no common eigenvalues, and} \quad (3.49)$$

$$\text{the pair } (A, \overline{C}^*) \quad \text{is stabilizable (in continuous time sense)} \quad (3.50)$$

where $\overline{C} = CP + DB^*$ and P is the solution of the Lyapunov-equation

$$AP + PA^* + BB^* = 0, \quad (3.51)$$

then the reachability subspaces of the given realizations of F and $F_{\mathbf{r}}$ coincide, i.e.

$$\langle A \mid B \rangle = \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle \quad (3.52)$$

(iv) if the reachability subspaces above coincide then the finite zero matrix of $F_{\mathbf{r}}$ (will be denoted by $\Lambda_f(F_{\mathbf{r}})$) is given by

$$\Lambda_f(F_{\mathbf{r}}) = \begin{bmatrix} -\sigma^{-1} (\Lambda_k + \alpha_k \beta_k)^* \sigma & \Lambda_{kf} + \sigma^{-1} (\beta_k^* R_0^* + H_k^*) H_f \\ 0 & \Lambda_f \end{bmatrix}. \quad (3.53)$$

Let us observe that the theorem shows that as a result of factoring out $\mathcal{W}(\ker F)$ the virtual zeros of F are materialized as *finite zeros* of $F_{\mathbf{r}}$ appearing on the right half plane of \mathbb{C} together with preserving the original finite zeros of F .

PROOF. (i) Let us first compute a realization of $F_{\mathbf{r}}$.

$$\begin{aligned} F_{\mathbf{r}}(z) &= F(z) L_{\beta}(z) \\ &= (D + C(zI - A)^{-1} B) (L_0 - (H_k + R_0 \beta_k) (zI - (\Lambda_k + \alpha_k \beta_k))^{-1} \sigma^{-1} H_k^* L_0) \\ &= DL_0 + C(zI - A)^{-1} (BL_0 + \Pi_k \sigma^{-1} H_k^* L_0), \end{aligned} \quad (3.54)$$

using the identities

$$B(H_k + R_0 \beta_k) = (zI - A) \Pi_k - \Pi_k (zI - \Lambda_k - \alpha_k \beta_k)$$

and

$$C \Pi_k + D(H_k + R_0 \beta_k) = 0,$$

proving part (i)

(ii) According to Lemma 3.1 to characterize the space $\mathcal{V}^*(\Sigma_{\mathbf{r}})$ a maximal solution of equation (3.10) should be considered. To this aim compute the following product:

$$\begin{bmatrix} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ C & DL_0 \end{bmatrix} \begin{bmatrix} \Pi_{\max} \\ L_0^* H_{\max} \end{bmatrix}.$$

Let us take the first element:

$$\begin{aligned}
A\Pi_{\max} + (B + \Pi_k\sigma^{-1}H_k^*)L_0L_0^*H_{\max} \\
&= \Pi_{\max}\Lambda_{\max} - BH_{\max} + BL_0L_0^*H_{\max} + \Pi_k\sigma^{-1}H_k^*L_0L_0^*H_{\max} \\
&= \Pi_{\max}\Lambda_{\max} - BR_0R_0^*H_{\max} + \Pi_k\sigma^{-1}H_k^*L_0L_0^*H_{\max} \\
&= \Pi_{\max}\Lambda_{\max} - \Pi_k\alpha_kR_0^*H_{\max} + \Pi_k\sigma^{-1}H_k^*L_0L_0^*H_{\max} . \quad (3.55)
\end{aligned}$$

On the other hand

$$C\Pi_{\max} + DL_0L_0^*H_{\max} = -DH_{\max} + D(I - R_0R_0^*)H_{\max} = 0 ,$$

proving that equation

$$\begin{bmatrix} A & (B + \Pi_k\sigma^{-1}H_k^*)L_0 \\ C & DL_0 \end{bmatrix} \begin{bmatrix} \Pi_{\max} \\ L_0^*H_{\max} \end{bmatrix} = \begin{bmatrix} \Pi_{\max}\Lambda_e \\ 0 \end{bmatrix} \quad (3.56)$$

holds, for some matrix Λ_e using that $\text{Im}\Pi_k \subset \text{Im}\Pi_{\max}$ and thus proving that $\text{Im}\Pi_{\max} \subset \mathcal{V}^*(\Sigma_{\mathbf{r}})$.

To prove the converse inclusion let us assume that the matrices $\bar{\Pi}$, \bar{H} , $\bar{\Lambda}$ provide a maximal solution of the equation

$$\begin{bmatrix} A & (B + \Pi_k\sigma^{-1}H_k^*)L_0 \\ C & DL_0 \end{bmatrix} \begin{bmatrix} \bar{\Pi} \\ \bar{H} \end{bmatrix} = \begin{bmatrix} \bar{\Pi}\bar{\Lambda} \\ 0 \end{bmatrix} . \quad (3.57)$$

Due to the maximality we already have that $\text{Im}\Pi_k \subset \text{Im}\Pi_{fzk} \subset \text{Im}\Pi$. Rearranging the terms in (3.57) we get that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{\Pi} \\ L_0\bar{H} \end{bmatrix} = \begin{bmatrix} \bar{\Pi}\bar{\Lambda} - \Pi_k\sigma^{-1}H_k^*L_0^*\bar{H} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{\Pi}\bar{\Lambda}' \\ 0 \end{bmatrix}$$

(where $\bar{\Lambda}'$ defined in an obvious way), giving that

$$\text{Im}(\bar{\Pi}) \subset \text{Im}(\Pi_{\max}) ,$$

thus

$$\mathcal{V}^*(\Sigma_{\mathbf{r}}) = \text{Im}(\bar{\Pi}) = \text{Im}(\Pi_{\max}) = \mathcal{V}^*(\Sigma) ,$$

proving (3.45).

To prove that $\mathcal{V}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma_{\mathbf{r}}) = \{0\}$ Corollary 3.6 can be applied giving that solutions of

$$\begin{bmatrix} (B + \Pi_k\sigma^{-1}H_k^*)L_0 \\ DL_0 \end{bmatrix} \xi = \begin{bmatrix} \bar{\Pi}\eta \\ 0 \end{bmatrix}$$

should be considered. Rearranging the first equation we obtain that

$$BL_0\xi = \bar{\Pi}\eta - \Pi_k\sigma^{-1}H_k^*L_0\xi \in \text{Im}\bar{\Pi} = \text{Im}\Pi_{\max} .$$

Using the second equation: $D(L_0\xi) = 0$, the maximality of R_0 in (3.21) gives that $L_0\xi \in \text{Im}(R_0)$. This implies that $\xi = 0$, consequently $\eta = 0$. Thus

$$\mathcal{V}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma_{\mathbf{r}}) = \{0\} ,$$

or in other words the module $\mathcal{W}(\ker F_{\mathbf{r}})$ is trivial.

To prove (3.47) we first verify the inclusion $\mathcal{C}^*(\Sigma_{\mathbf{r}}) \subset \mathcal{C}^*(\Sigma)$. Since the elements of $\mathcal{C}^*(\Sigma_{\mathbf{r}})$ are those vectors in the state-space which are reachable from the origin via a trajectory producing no output we might apply an induction argument. Obviously, $0 \in \mathcal{C}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma)$.

Now, if $\xi \in \mathcal{C}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma)$ and equations

$$\begin{aligned} \eta &= A\xi + (BL_0 + \Pi_k \sigma^{-1} H_k^* L_0) u \\ 0 &= C\xi + DL_0 u \end{aligned}$$

hold, then $\eta \in \mathcal{C}^*(\Sigma_{\mathbf{r}})$ and $A\xi + BL_0 u \in \mathcal{C}^*(\Sigma)$, while $\Pi_k \sigma^{-1} H_k^* L_0 u \in \text{Im} \Pi_k = \mathcal{V}^*(\Sigma) \cap \mathcal{C}^*(\Sigma)$. Thus $\eta \in \mathcal{C}^*(\Sigma)$, as well.

By induction this proves that

$$\mathcal{C}^*(\Sigma_{\mathbf{r}}) \subset \mathcal{C}^*(\Sigma) .$$

Conversely, if $\xi \in \mathcal{C}^*(\Sigma) \cap (\mathcal{C}^*(\Sigma_{\mathbf{r}}) \vee \text{Im} \Pi_k)$ and

$$\begin{aligned} \eta &= A\xi + Bu \\ 0 &= C\xi + Du \end{aligned}$$

then $\eta \in \mathcal{C}^*(\Sigma)$. Introducing the notation $u_1 = R_0^* u$, $u_2 = L_0^* u$ we get that $u = R_0 u_1 + L_0 u_2$.

The assumption $\xi \in (\mathcal{C}^*(\Sigma_{\mathbf{r}}) \vee \text{Im} \Pi_k)$ implies that

$$\xi = \xi_1 + \Pi_k v ,$$

for some vectors ξ_1, v , where $\xi_1 \in \mathcal{C}^*(\Sigma_{\mathbf{r}})$. Now

$$\begin{aligned} C\xi_1 + D(L_0 u_2 - H_k v) &= C\xi_1 + DL_0 u_2 - DH_k v + DR_0 R_0^* H_k v \\ &= C\xi_1 + DL_0 u_2 + C\Pi_k v \\ &= C\xi + Du = 0 \end{aligned}$$

implying that

$$A\xi_1 + (B + \Pi_k \sigma^{-1} H_k^*) L_0 (u_2 - L_0^* H_k v) \in \mathcal{C}^*(\Sigma_{\mathbf{r}}) .$$

On the other hand

$$\begin{aligned} \eta &= A\xi + Bu \\ &= A\xi_1 + A\Pi_k v + BR_0 u_1 + BL_0 u_2 \\ &= A\xi_1 + (B + \Pi_k \sigma^{-1} H_k^*) L_0 (u_2 - L_0^* H_k v) - \Pi_k \sigma^{-1} H_k^* L_0 (u_2 - L_0^* H_1 v) \\ &\quad + BL_0 L_0^* H_k v + A\Pi_k v + BR_0 u_1 \\ &= A\xi_1 + (B + \Pi_k \sigma^{-1} H_k^*) L_0 (u_2 - L_0^* H_k v) \\ &\quad - \Pi_k \sigma^{-1} H_k^* L_0 (u_2 - L_0^* H_k v) + \Pi_k \Lambda_k v + \Pi_k \alpha_k (u_2 - R_0^* H_k v) . \end{aligned}$$

thus

$$\eta \in (\mathcal{C}^*(\Sigma_{\mathbf{r}}) \vee \text{Im}(\Pi_k)) .$$

Induction argument gives that

$$\mathcal{C}^*(\Sigma) \subset (\mathcal{C}^*(\Sigma_{\mathbf{r}}) \vee \text{Im}\Pi_k)$$

Consequently

$$\mathcal{C}^*(\Sigma) = \text{Im}\Pi_k \vee \mathcal{C}^*(\Sigma_{\mathbf{r}}) = (\mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)) \vee \mathcal{C}^*(\Sigma_{\mathbf{r}}) .$$

Finally the identity (3.45) and Corollary 3.6 imply that

$$\mathcal{W}(\ker F_{\mathbf{r}}) = \{0\} ,$$

concluding the proof of part (ii).

(iii) To prove the first part let us recall that $\text{Im}(\Pi_k) \subset \text{Im}(\Pi_{fzk}) = \mathcal{V}^*(\sigma) \cap \langle A \mid B \rangle$. Thus if the column-vectors of the matrix ξ form a basis in the orthogonal complement of the reachability subspace of $\langle A \mid B \rangle$ i.e. $\xi^*B = 0$ and $\xi^*A = \kappa\xi^*$ holds for some matrix κ then $\xi^*\Pi_k = 0$, as well. Consequently, $\xi^*(B + \Pi_k\sigma^{-1}H_k^*)L_0 = 0$ giving that the columns of ξ are orthogonal to the elements of $\langle A \mid (B + \Pi_k\sigma^{-1}H_k^*)L_0 \rangle$ proving the first inclusion.

To prove the second part of (iii) let us first consider two identities.

$$\begin{aligned} B &= (B + \Pi_k\sigma^{-1}H_k^*)(L_0L_0^* + R_0R_0^*) - \Pi_k\sigma^{-1}H_k^* \\ &= (B + \Pi_k\sigma^{-1}H_k^*)L_0L_0^* + \Pi_k(\sigma^{-1}H_k^*R_0R_0^* + \alpha_kR_0^* - \sigma^{-1}H_k^*) \\ &= (B + \Pi_k\sigma^{-1}H_k^*)L_0L_0^* - \Pi_k\sigma^{-1}(\beta_k^*R_0^* + H_k^*) \end{aligned} \quad (3.58)$$

using that $BR_0 = \Pi_k\alpha_k$, and (3.37) and

$$\begin{aligned} A\Pi_k + (B + \Pi_k\sigma^{-1}H_k^*)L_0L_0^*H_k &= A\Pi_k + BH_k + \Pi_k\sigma^{-1}(\beta_k^*R_0^* + H_k^*)H_k \\ &= \Pi_k\Lambda_k + \Pi_k\sigma^{-1}(\beta_k^*R_0^* + H_k^*)H_k \\ &= \Pi_k(\Lambda_k + \sigma^{-1}(\beta_k^*R_0^* + H_k^*)H_k) \\ &= -\Pi_k\sigma^{-1}(\Lambda_k + \alpha_k\beta_k)^*\sigma , \end{aligned} \quad (3.59)$$

using that $A\Pi_k + BH_k = \Pi_k\Lambda_k$ and equations (3.34), (3.35).

Consider first the assumption formulated in a), i.e. if all the eigenvalues of the matrix A have non-positive real part then the reachability subspaces of the given realizations of F and $F_{\mathbf{r}}$ coincide.

If the column-vectors of the matrix ξ form a basis in the orthogonal complement of the reachability subspace of the realization above of $F_{\mathbf{r}}$, then $\xi^*A = \kappa\xi^*$ for some matrix κ and $\xi^*(B + \Pi_k\sigma^{-1}H_k^*)L_0 = 0$. The eigenvalues of the matrix κ should form a subset of those of A .

Equations (3.59) and (3.58) imply that $\kappa\xi^*\Pi_k = -\xi^*\Pi_k\sigma^{-1}(\Lambda_k + \alpha_k\beta_k)^*\sigma$ showing in particular that $\text{Im}(\xi^*\Pi_k)$ is κ -invariant. Since according to the proof of Theorem 3.6 the matrix $\Lambda_k + \alpha_k\beta_k$ is asymptotically stable we get that on the subspace $\text{Im}(\xi^*\Pi_k)$ the eigenvalues of the matrix κ have positive real part.

But according to the assumption the spectrum of the matrix A is in the closed left half plane, consequently the eigenvalues of κ should have non-positive real part. Thus the

equation $\xi^* \Pi_k = 0$ holds true implying that $\xi^* B = 0$, as well. So the columns of ξ are orthogonal to the reachability subspace of the realization of F . I.e.

$$\langle A \mid B \rangle \subset \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle ,$$

proving the second part of (iii) using the assumption formulated in a).

To prove the converse inclusion based on the assumption b) assume again that the column-vectors of the matrix ξ form a basis in the orthogonal complement of the reachability subspace of the realization above of F_r , then $\xi^* A = \kappa \xi^*$ for some matrix κ and $\xi^* (B + \Pi_k \sigma^{-1} H_k^*) L_0 = 0$. Now from (3.58) we get that

$$\xi^* B = -\xi^* \Pi_k \sigma^{-1} (H_k + R_0 \beta_k)^* .$$

Multiplying from the right by D^* and using equation (3.32) and (3.31) we obtain that

$$\xi^* B D^* = \xi^* \Pi_k \sigma^{-1} \Pi_k^* C^* .$$

On the other hand multiplying the Lyapunov-equation (3.51) above from the left by ξ^* and using again equations (3.32), (3.31) and (3.59) we arrive at the following equation

$$\begin{aligned} -\xi^* P A^* &= \xi^* A P + \xi^* B B^* \\ &= \kappa \xi^* P - \xi^* \Pi_k \sigma^{-1} (H_k + R_0 \beta_k)^* B^* \\ &= \kappa \xi^* P - \xi^* \Pi_k \sigma^{-1} ((\Lambda_k + \alpha_k \beta_k)^* \Pi_k^* - \Pi_k^* A^*) \\ &= \kappa \xi^* P + \kappa \xi^* \Pi_k \sigma^{-1} \Pi_k^* + \xi^* \Pi_k \sigma^{-1} \Pi_k^* A^* . \end{aligned}$$

Rearranging it

$$\kappa \xi^* (P + \Pi_k \sigma^{-1} \Pi_k^*) = -\xi^* (P + \Pi_k \sigma^{-1} \Pi_k^*) A^* .$$

Since according to our assumption the spectra of A and $-A^*$ are disjoint but the spectrum of κ should be a subset of that of A we find that

$$\xi^* (P + \Pi_k \sigma^{-1} \Pi_k^*) = 0 .$$

Consequently,

$$\xi^* \overline{C}^* = \xi^* (P C^* + B D^*) = \xi^* (-\Pi_k \sigma^{-1} \Pi_k^* C^* + \Pi_k \sigma^{-1} \Pi_k^* C^*) = 0 .$$

Thus the eigenvalues of κ belong to the uncontrollable (with respect to the pair (A, \overline{C}^*)) eigenvalues of A . According to the assumption these eigenvalues have non-positive real part, but equation (3.59) implies that on the subspace $\text{Im}(\xi^* \Pi_k)$ the matrix $(-\kappa)$ should be asymptotically stable. Thus $\xi^* \Pi_k = 0$. Consequently,

$$\xi^* B = -\xi^* \Pi_k \sigma^{-1} (H_k + R_0 \beta_k)^* = 0 .$$

Thus

$$\langle A \mid B \rangle \subset \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle ,$$

proving in this case, as well, that these two reachability subspaces coincide.

(iv) To conclude the proof of the theorem the finite zero matrix of F_r should be computed. According to Theorem 3.4 equation (3.46) gives that $\mathcal{W}(\ker F_r) = \{0\}$, consequently from Corollary 3.3 it follows that to identify the finite zero matrix of F_r a basis in $\mathcal{V}^*(\Sigma_r) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle$ should be considered and taken as the matrix “ Π ” in the corresponding form of the equation (3.15). Now equation (3.45) in part (ii) and the assumption concerning the reachability subspaces imply that

$$\mathcal{V}^*(\Sigma_r) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle = \mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle$$

so the columns of Π_{fzk} form a basis in this subspace. Thus it is reasonable to compute the product

$$\begin{bmatrix} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ C & DL_0 \end{bmatrix} \begin{bmatrix} \Pi_{fzk} \\ L_0^* H_{fzk} \end{bmatrix}. \quad (3.60)$$

Let us take the first element:

$$\begin{aligned} A \Pi_{fzk} + (B + \Pi_k \sigma^{-1} H_k^*) L_0 L_0^* H_{fzk} \\ &= \Pi_{fzk} \Lambda_{fzk} - B H_{fzk} + B L_0 L_0^* H_{fzk} + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_{fzk} \\ &= \Pi_{fzk} \Lambda_{fzk} - B R_0 R_0^* H_{fzk} + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_{fzk} \\ &= \Pi_{fzk} \Lambda_{fzk} - \Pi_k \alpha_k R_0^* H_{fzk} + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_{fzk}. \end{aligned} \quad (3.61)$$

Taking the partitioned form of these matrices we obtain that the first block is

$$\begin{aligned} \Pi_k \Lambda_k &- \Pi_k \alpha_k R_0^* H_k + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_k \\ &= \Pi_k \sigma^{-1} (\sigma (\Lambda_k - \alpha_k R_0^* H_k) + H_k^* (I - R_0 R_0^*) H_k) \\ &= -\Pi_k \sigma^{-1} ((\Lambda_k - \alpha_k R_0^* H_k)^* \sigma - \sigma \alpha_k \alpha_k^* \sigma) \\ &= -\Pi_k \sigma^{-1} (\Lambda_k + \alpha_k \beta_k)^* \sigma, \end{aligned}$$

and also

$$\Pi_k \Lambda_k - \Pi_k \alpha_k R_0^* H_k + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_k = \Pi_k (\Lambda_k + \sigma^{-1} (\beta_k^* R_0^* + H_k^*) H_k)$$

using the Riccati-equation (3.38) and the identity (3.37).

Let us compute the second block:

$$\begin{aligned} \Pi_k \Lambda_{kf} + \Pi_f \Lambda_f - \Pi_k \alpha_k R_0^* H_f + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* H_f \\ &= \Pi_k (\Lambda_{kf} + \sigma^{-1} (\beta_k^* R_0^* + H_k^*) H_f) + \Pi_f \Lambda_f. \end{aligned} \quad (3.62)$$

On the other hand the second element in (3.60):

$$C \Pi_{fzk} + D L_0 L_0^* H_{fzk} = -D H_{fzk} + D (I - R_0 R_0^*) H_{fzk} = 0,$$

proving that equation

$$\begin{bmatrix} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ C & DL_0 \end{bmatrix} \begin{bmatrix} \Pi_{fzk} \\ L_0^* H_{fzk} \end{bmatrix} = \begin{bmatrix} \Pi_{fzk} (\Lambda_{fzk} + \Gamma H_{fzk}) \\ 0 \end{bmatrix} \quad (3.63)$$

holds, where

$$\Gamma = \begin{bmatrix} \sigma^{-1}(\beta_k^* R_0^* + H_k^*) \\ 0 \end{bmatrix}. \quad (3.64)$$

Thus – using Corollary 3.3 and $\mathcal{W}(\ker F_{\mathbf{r}}) = \{0\}$ from part (ii) – the matrix

$$\Lambda_{fzk} + \Gamma H_{fzk} = \begin{bmatrix} -\sigma^{-1}(\Lambda_k + \alpha_k \beta_k)^* \sigma & \Lambda_{kf} + \sigma^{-1}(\beta_k^* R_0^* + H_k^*) H_f \\ 0 & \Lambda_f \end{bmatrix}$$

is the finite zero matrix of $F_{\mathbf{r}}$, concluding the proof of (iv) and that of the theorem. \blacksquare

REMARK 3.13 Let us note that the function $F_{\mathbf{r}}$ is *left-invertible*. In fact, according to the Remark 3.5 and equation (3.46) it remains only to check the kernel of $\begin{bmatrix} (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ DL_0 \end{bmatrix}$. Now if for some vector ξ the identity $(B + \Pi_k \sigma^{-1} H_k^*) L_0 \xi = 0$ holds, then obviously $BL_0 \xi \in \text{Im} \Pi_k \subset \text{Im} \Pi$. If moreover $DL_0 \xi = 0$, as well, then – using the maximality of $R_0 - L_0 \xi \in \text{Im} R_0$ should hold. But this implies that $L_0 \xi = 0$, so $\xi = 0$. I.e. both conditions for the left-invertibility hold.

REMARK 3.14 Let us point out that even in the case when there is a reduction in the reachability subspace the finite zero matrix Λ_f of F appears in the finite zero matrix of $F_{\mathbf{r}}$.

In fact, we are going to show that

$$\begin{aligned} \dim [\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle] &= \dim [\mathcal{V}^*(\Sigma_{\mathbf{r}}) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle] \\ &= \dim (\text{Im} \Pi_k) - \dim [\text{Im}(\Pi_k) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle] \end{aligned} \quad (3.65)$$

i.e. the "reduction" affects only the subspace $\mathcal{C}^*(\sigma) \cap \mathcal{V}^*(\sigma) = \mathcal{R}^*(\sigma)$

Let us observe that the inclusion $\text{Im}(\Pi_k) \subset \mathcal{V}^*(\Sigma) = \mathcal{V}^*(\Sigma_{\mathbf{r}})$ implies that the inequality \geq holds trivially.

To prove the converse inequality let us consider a matrix ξ with columns forming a basis in the orthogonal complement of the reachability subspace $\langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle$. Then

$$\text{rank } \xi^* \Pi_k = \dim (\text{Im} \Pi_k) - \dim [\text{Im}(\Pi_k) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle].$$

We are going to show that the inclusions $\text{Im} [\xi^* A^j B] \subset \text{Im} \xi^* \Pi_k$ hold, for all $j \geq 0$ proving that $\text{rank } \xi^* [B, AB, A^2 B, \dots] \leq \text{rank} \xi^* \Pi_k$. In fact, equation (3.58) gives that $\xi^* B = \xi^* \Pi_k \sigma^{-1} (\beta_k^* R_0^* + H_k^*)$, thus $\text{Im} \xi^* B \subset \text{Im} \xi^* \Pi_k$. On the other hand (3.59) gives immediately that $\text{Im} \xi^* A \Pi_k \subset \text{Im} \xi^* \Pi_k$. Starting from these observation we shall prove by induction that $\text{Im} \xi^* A^j \Pi_k \subset \text{Im} \xi^* \Pi_k$ and $\text{Im} \xi^* A^j B \subset \text{Im} \xi^* \Pi_k$ for all $j \geq 0$.

Equations (3.59) and (3.58) imply that

$$\begin{aligned} \xi^* AB &= -\xi^* A \Pi_k \sigma^{-1} (\beta_k^* R_0^* + H_k^*) \\ &= \xi^* \Pi_k \sigma^{-1} (\Lambda_k + \alpha_k \beta_k)^* (\beta_k^* R_0^* + H_k^*) \end{aligned}$$

Using the equations (3.58) and $A \Pi_k = -B H_k + \Pi_k \Lambda_k$ we can write

$$\begin{aligned} \xi^* A^j B &= -\xi^* A^j \Pi_k \sigma^{-1} (\beta_k^* R_0^* + H_k^*) \\ &= -\xi^* A^{j-1} (-B H_k + \Pi_k \Lambda_k) \sigma^{-1} (\beta_k^* R_0^* + H_k^*) . \end{aligned}$$

Thus $\text{Im } \xi^* A^j B \subset \text{Im } \xi^* A^j \Pi_k \subset \text{Im } \xi^* A^{j-1} B \vee \text{Im } \xi^* A^{j-1} \Pi_k \subset \text{Im } \xi^* \Pi_k$ by induction.
Consequently,

$$\begin{aligned} \dim [\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle] &= \dim [\mathcal{V}^*(\Sigma_r) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle] \\ &= \dim [\mathcal{V}^*(\Sigma) \cap \langle A \mid B \rangle] - \dim [\mathcal{V}^*(\Sigma) \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle] \\ &\leq \dim \langle A \mid B \rangle - \dim \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle \leq \text{rank } \xi^* \Pi_k \\ &= \dim(\text{Im } \Pi_k) - \dim [\text{Im } \Pi_k \cap \langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle] , \end{aligned}$$

proving the converse inequality, as well.

3.2 The zero module $\mathcal{W}(\text{Im } F)$

Now let us turn to the analysis of the space

$$\mathcal{W}(\text{Im } F) = \frac{\pi_-(\text{Im } F)}{\text{Im } F \cap z^{-1} \Omega_\infty Y} .$$

A p -tuple h is in $\pi_-(\text{Im } F)$ if it is strictly proper and there exists a polynomial p -tuple ϕ such that $h + \phi \in \text{Im}(F)$. Two such functions h_1, h_2 are considered to be equivalent if $h_1 - h_2 \in \text{Im}(F)$.

Based on these observations the following theorem gives a “state-space” characterization of the elements in $\mathcal{W}(\text{Im } F)$.

Theorem 3.8. *Assume that the pair (C, A) is observable. Then the equivalence classes of $\mathcal{W}(\text{Im } F)$ are determined by the functions*

$$C(zI - A)^{-1} \beta$$

where $\beta \in \langle A \mid B \rangle$ and two functions – given by the vectors β_1, β_2 – are considered to be equivalent if

$$\beta_1 - \beta_2 \in \mathcal{V}^*(\Sigma) \vee \mathcal{C}^*(\Sigma) .$$

PROOF. Consider a rational q -tuple $g(z)$. Assume that

$$g(z) = H_1(zI - \Lambda_1)^{-1} G_1 + g_0 + g_1 z + \cdots + g_k z^k .$$

Then the observation that $z^j(zI - A)^{-1} - A^j(zI - A)^{-1}$ is a polynomial implies that

$$\pi_-(F(z)g(z)) = F(z)H_1(zI - \Lambda_1)^{-1} G_1 + C(zI - A)^{-1} \sum_{l=0}^k A^l B g_l .$$

The first term is strictly proper and in the space $\text{Im } F$ thus the second term determines the corresponding equivalence class. By definition any function of the form $C(zI - A)^{-1} \beta$ where $\beta \in \langle A \mid B \rangle$ can be obtained this way. But possibly different β vectors might generate the same equivalence class.

Thus we have to characterize those $\beta \in \langle A \mid B \rangle$ vectors for which $C(zI - A)^{-1}\beta \in \text{Im}F \cap z^{-1}\Omega_\infty Y$. To this aim assume that $Fg = C(zI - A)^{-1}\beta$ is strictly proper for some rational function g with the form given above.

Straightforward calculation gives the polynomial part of the product. Namely it is

$$Dg_k z^k + \sum_{j=0}^{k-1} \left(Dg_j + \sum_{l=0}^{k-1-j} CA^l Bg_{l+j+1} \right) z^j .$$

This should be zero. Since the polynomial part of g gives rise to $C(zI - A)^{-1} \sum_{l=0}^k A^l Bg_l$ in the strictly proper part of Fg , we get that

$$\beta_1 = \sum_{l=0}^k A^l Bg_l$$

should be an output-nulling reachable element, or in other words $\beta_1 \in \mathcal{C}^*(\Sigma)$. Introducing the notation $\beta_2 = \beta - \beta_1$, we obtain that $F(z)H_1(zI - \Lambda_1)^{-1}G_1 = C(zI - A)^{-1}\beta_2$. Using the observability of the pair (C, A) and Proposition 3.1 we obtain that

$$\beta_2 = \Pi_1 G_1 ,$$

where Π_1 is a solution of the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \Pi_1 \Lambda_1 \\ 0 \end{bmatrix} .$$

I.e. β_2 is in the maximal output-nulling controlled invariant set, $\beta_2 \in \mathcal{V}^*(\Sigma)$.

Conversely, if $\beta \in \langle A \mid B \rangle$, and $\beta = \beta_1 + \beta_2$, $\beta_1 \in \mathcal{C}^*(\Sigma) \subset \langle A \mid B \rangle$, $\beta_2 \in \mathcal{V}^*(\Sigma)$, then β_1 can be written in the form

$$\beta_1 = \sum_{l=0}^k A^l Bg_l$$

in such a way that for the polynomial $g(z) = \sum_{l=0}^k g_l z^l$ the identity

$$F(z)g(z) = C(zI - A)^{-1}\beta_1$$

holds true. On the other hand assume that the triplet $(\Pi_{\max}, H_{\max}, \Lambda_{\max})$ forms a maximal solution of (3.10). Then $\beta_2 \in \mathcal{V}^*(\Sigma)$ implies that

$$\beta_2 = -\Pi_{\max} G$$

for some vector G . Now immediate calculation gives that

$$F(z)H_{\max}(zI - \Lambda_{\max})^{-1}G = -C(zI - A)^{-1}\Pi_{\max}G = C(zI - A)^{-1}\beta_2 .$$

Consequently,

$$F(z)(H_{\max}(zI - \Lambda_{\max})^{-1}G + g(z)) = C(zI - A)^{-1}\beta ,$$

thus it is in the space $\text{Im}F \cap z^{-1}\Omega_\infty Y$, concluding the proof of the theorem. ■

REMARK 3.15 The identification of the co-range of the function F to the factor-space $\langle A \mid B \rangle / ((\langle A \mid B \rangle \cap (\mathcal{V}^*(\Sigma) \vee \mathcal{C}^*(\Sigma)))$ can be found e.g. in [1] (even without the assumption of the observability of (C, A)) but without explicitly identifying the functions in the equivalence classes of $\mathcal{W}(\text{Im } F)$.

REMARK 3.16 Assume that the pair (C, A) is observable, and the eigenvalues of A are in the closed left half plane or – more generally – conditions (3.49 and (3.50) hold. Consider the function F_r defined in (3.43). Due to the fact that it has the same " (C, A) " pair as the function F , the previous theorem together with part (ii) and (iii) of Theorem 3.7 imply that

$$\mathcal{W}(\text{Im } F_r) = \mathcal{W}(\text{Im } F) .$$

3.3 Zeros at infinity

Let us recall the definition of the zero module at infinity:

$$Z_\infty(F) = \frac{F^{-1}(z^{-1}\Omega_\infty Y) + z^{-1}\Omega_\infty U}{\ker F + z^{-1}\Omega_\infty U} .$$

I.e. the q -tuples of rational functions g should be considered for which there exist a strictly proper rational q -tuple h such that

$$F(g + h) \text{ is strictly proper,} \quad (3.66)$$

and g_1, g_2 with this property are considered to be equivalent if for some strictly proper q -tuple h the identity

$$F(g_1 - g_2 + h) = 0 .$$

Theorem 3.9. *Assume that the pair (C, A) is observable. Then the equivalence classes in $Z_\infty(F)$ are determined by the vectors in $\mathcal{C}^*(\Sigma)$ in the sense that for any $\beta \in \mathcal{C}^*(\Sigma)$ there exists a finite input sequence producing no output but giving β as the next immediate state-vector. The input sequence gives the coefficients of a polynomial in $F^{-1}(z^{-1}\Omega_\infty Y) + z^{-1}\Omega_\infty U$.*

Two polynomials are taken to be equivalent if the difference of the corresponding β vectors are in $\mathcal{R}^(\Sigma) = \mathcal{V}^*(\Sigma) \cap \mathcal{C}^*(\Sigma) = \text{Im}(\Pi_k)$, see (3.33).*

PROOF. Since F is assumed to be proper the function Fh is strictly proper if h is strictly proper. Thus the condition in (3.66) states that Fg should be strictly proper. Due to our assumption that the function F is proper this is equivalent to

$$\pi_+(F\pi_+(g)) = 0 .$$

Using the notation

$$\pi_+(g) = g_0 + g_1 z + \dots g_k z^k ,$$

we get that the sequence g_k, g_{k-1}, \dots, g_0 gives an output-nulling input sequence, so it takes the origin into some state-vector $\beta \in \mathcal{C}^*(\Sigma)$.

Two such sequences are considered to be equivalent if adding to their difference an appropriate strictly proper function a function in $\ker F$ is obtained. So let us assume that

$$g(z) = H_1 (zI - \Lambda_1)^{-1} G_1 + g_0 + g_1 z + \cdots + g_k z^k \in \ker F .$$

Under the assumption that the input sequence g_k, g_{k-1}, \dots, g_0 produces no output we get that the polynomial part of the product Fg is zero. Thus, computing the strictly proper part of Fg the equation

$$(D + C(zI - A)^{-1} B) H_1 (zI - \Lambda_1)^{-1} G_1 + C(zI - A)^{-1} \sum_{j=0}^k A^j B g_j = 0$$

is obtained. Proposition 3.1 implies that there exists a matrix Π_1 such that equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} \Pi_1 \Lambda_1 \\ 0 \end{bmatrix} , \quad \Pi_1 G_1 = \sum_{j=0}^k A^j B g_j$$

hold.

Conversely, if

$$\beta = \sum_{j=0}^k A^j B g_j \in \mathcal{V}^*(\Sigma) ,$$

for an output-nulling input sequence g_k, g_{k-1}, \dots, g_0 then there exists a vector G such that $\beta = \Pi_{\max} G$. Using the identity

$$F(z) H_{\max} (zI - \Lambda_{\max})^{-1} = -C(zI - A)^{-1} \Pi_{\max}$$

straightforward computation gives that

$$H_{\max} (zI - \Lambda_{\max})^{-1} G + \sum_{j=0}^k g_j z^j \in \ker F .$$

Thus the polynomial $g_0 + g_1 z + \cdots + g_k z^k$ (with output-nulling input sequence coefficients) is considered to be equivalent to zero if and only if the state vector $\beta = \sum_{j=0}^k A^j B g_j$ is in $\text{Im } \Pi_{\max} = \mathcal{V}^*(\Sigma)$. I.e. $\beta \in \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)$. ■

REMARK 3.17 Again this Theorem should be compared to Theorem 4 in [1].

Corollary 3.8. *Assume that the pair (C, A) is observable. Then the subspace $Z_{\infty}(F)$ is trivial if and only if*

$$\{B\eta \mid D\eta = 0\} \subset \mathcal{V}^*(\Sigma) = \text{Im}(\Pi_{\max}) . \quad (3.67)$$

PROOF. The previous theorem implies that $Z_{\infty}(F)$ is trivial if and only if $\mathcal{C}^*(\Sigma) \subset \mathcal{V}^*(\Sigma)$. Since the set $\mathcal{C}^*(\Sigma)$ contains those vectors which are reachable from the origin with zero output, and the set $\{B\eta \mid D\eta = 0\}$ contains those vectors which can be reached from the

origin in one step with zero output, we obtain that if $Z_\infty(F)$ is trivial then $\{B\eta \mid D\eta = 0\} \subset \mathcal{V}^*(\Sigma)$.

Conversely, assume that $\{B\eta \mid D\eta = 0\} \subset \mathcal{V}^*(\Sigma)$. We show by induction that in this case $\mathcal{C}^*(\Sigma) \subset \mathcal{V}^*(\Sigma)$. Consider a maximal solution $(\Pi_{\max}, H_{\max}, \Lambda_{\max})$ of (3.10). According to Lemma 3.1 $\mathcal{V}^*(\Sigma) = \text{Im}(\Pi_{\max})$. Assume that $x \in \text{Im}(\Pi_{\max})$, i.e. $x = \Pi_{\max}\xi$ for some ξ , and equations

$$\begin{aligned} x_+ &= Ax + Bu \\ 0 &= Cx + Du \end{aligned}$$

hold true. Equation (3.10) gives that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{\max}\xi \\ H_{\max}\xi \end{bmatrix} = \begin{bmatrix} \Pi_{\max}\Lambda_{\max}\xi \\ 0 \end{bmatrix}.$$

Taking the difference

$$\begin{bmatrix} B \\ D \end{bmatrix} (u - H_{\max}\xi) = \begin{bmatrix} x_+ - \Pi_{\max}\Lambda_{\max}\xi \\ 0 \end{bmatrix}.$$

The assumption implies that $x_+ - \Pi_{\max}\Lambda_{\max}\xi = B(u - H_{\max}\xi) \in \text{Im}(\Pi_{\max})$, giving that $x_+ \in \text{Im}(\Pi_{\max})$ and concluding the proof of the corollary. \blacksquare

REMARK 3.18 Assume that the pair (C, A) is observable. Consider the function $F_{\mathbf{r}}$ defined in (3.43). The previous theorem together with part (ii) of Theorem 3.7 implies that

$$Z_\infty(F_{\mathbf{r}}) = Z_\infty(F).$$

3.4 Zero modules of F vs. $F_{\mathbf{r}}$

It is worth summarizing the connections between the various zero modules of F and $F_{\mathbf{r}}$. This is the subject of the next proposition.

Proposition 3.5. *Assume that F has the realization*

$$F(z) \sim \left(\frac{A \mid B}{C \mid D} \right)$$

where (C, A) is an observable pair.

Then the function F has the following factorization

$$F = F_{\mathbf{r}} L_\beta^*$$

where L_β is a tall inner function, and

(i)

$$\mathcal{W}(\ker F_{\mathbf{r}}) = \{0\} ;$$

(ii)

$$Z_{\infty}(F_{\mathbf{r}}) = Z_{\infty}(F) ;$$

(iii) if all the eigenvalues of A are in the closed left half-plane or conditions (3.49) and (3.50) hold then

the McMillan-degrees of F and $F_{\mathbf{r}}$ are equal.

(iv) if the McMillan-degrees of F and $F_{\mathbf{r}}$ are equal then

(a)

$$\mathcal{W}(\operatorname{Im} F_{\mathbf{r}}) = \mathcal{W}(\operatorname{Im} F)$$

and

(b) the finite zero matrix of $F_{\mathbf{r}}$ is given as (using the notation given in Theorem 3.7):

$$\Lambda_f(F_{\mathbf{r}}) = \begin{bmatrix} -\sigma^{-1}(\Lambda_k + \alpha_k \beta_k)^* \sigma & \Lambda_{kf} + \sigma^{-1}(\beta_k^* R_0^* + H_k^*) H_f \\ 0 & \Lambda_f \end{bmatrix}$$

i.e. the finite zero matrix Λ_f of F is extended.

4 Connections between the left and right zero module spaces

In the previous sections the zero module spaces were defined with respect to the transformation $h \rightarrow Fh$. For a fixed matrix valued rational function we might consider the left multiplication, i.e. $g \rightarrow gF$, and define the corresponding zero modules accordingly. The previous theorems and propositions can be carried over to cover this case almost without any changes.

For example – assuming that the realization of F provided by the matrices (A, B, C, D) is minimal – according to Corollary 3.2 to characterize the spaces $Z_{\text{left}}(F) \oplus \mathcal{W}(\ker_{\text{left}} F)$ (where the subtext “left” indicates that these spaces are defined with respect to the left multiplication) maximal solution of the equation

$$[\Pi'_{\max}, H'_{\max}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\Lambda'_{\max} \Pi'_{\max}, 0] \quad (4.68)$$

should be considered.

The following theorem connects various “left” and “right” subspaces.

Theorem 4.1. Assume that F has the realization

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) .$$

Consider a maximal solution of the equation (4.68). Then

$$\ker \Pi'_{\max} = \mathcal{C}^*(\Sigma) ,$$

in other words $(\mathcal{V}_{\text{left}}^*)^\perp(\Sigma) = \mathcal{C}^*(\Sigma)$, (with the obvious meaning of the notation $\mathcal{V}_{\text{left}}^*(\Sigma)$).

PROOF. First we show that $\ker \Pi'_{\max} \subset \mathcal{C}^*(\Sigma)$. To this aim we use the following well-known construction from geometric control theory: define recursively the following subspaces of row vectors

$$\mathcal{L}^r = \{z \mid \exists \eta \text{ such that } zA + \eta C \in \mathcal{L}^{r-1}, \text{ and } zB + \eta D = 0\} ,$$

($\mathcal{L}^0 = \mathbb{C}^n$.) Then $\mathcal{L}^r \subset \mathcal{L}^{(r-1)}$ and $\cap_r \mathcal{L}^r$ equals to the space spanned by the rows of Π'_{\max} .

We prove by induction that for any r the vectors orthogonal to the subspace \mathcal{L}^r are in the subspace $\mathcal{C}^*(\Sigma)$. Obviously,

$$\mathcal{L}^1 = \{z \mid \exists \eta : zB + \eta D = 0\} .$$

Now if the vector α is orthogonal to the elements of \mathcal{L}^1 then the equation

$$[z, \eta] \begin{bmatrix} B \\ D \end{bmatrix} = 0$$

implies that

$$[z, \eta] \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 0 .$$

Consequently, there exists a vector ζ such that

$$\begin{bmatrix} B \\ D \end{bmatrix} \zeta = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} .$$

In other words $\alpha = B\zeta$ can be reached from the origin in one step with zero output, thus $\alpha \in \mathcal{C}^*(\Sigma)$.

For each r consider a basis in \mathcal{L}^r and form the matrix Π^r containing the basis-vectors as its rows. Then

$$\mathcal{L}^r = \{z \mid \exists \eta, \lambda \text{ such that } zA + \eta C = \lambda \Pi^{(r-1)}, \text{ and } zB + \eta D = 0\} .$$

Assume that the vectors orthogonal to $\mathcal{L}^{(r-1)}$ are in the subspace $\mathcal{C}^*(\Sigma)$. Now if α is orthogonal to the elements of \mathcal{L}^r then the equation

$$[z, \eta, -\lambda] \begin{bmatrix} A & B \\ C & D \\ \Pi^{(r-1)} & 0 \end{bmatrix} = [0, 0]$$

should imply that

$$[z, \eta, -\lambda] \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = 0 .$$

Consequently, there exist ζ, ξ such that

$$\begin{bmatrix} A & B \\ C & D \\ \Pi^{(r-1)} & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \xi \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} .$$

In details, $\Pi^{(r-1)}\zeta = 0$ thus $\zeta \in \mathcal{C}^*(\Sigma)$. Also, $A\zeta + B\xi = \alpha$, $C\zeta + D\xi = 0$, so α can be reached from ζ in one step with zero output. The induction hypothesis gives that $\alpha \in \mathcal{C}^*(\Sigma)$, as well. The identity $\ker \Pi'_{\max} = \cup \ker \Pi^r$, and $\ker \Pi^r \supset \ker \Pi^{(r-1)}$ implies that

$$\ker \Pi'_{\max} \subset \mathcal{C}^*(\Sigma) .$$

Conversely, assume that the vector $\alpha \in \mathcal{C}^*(\Sigma)$. We are going to show that $\alpha \in \ker \Pi'$, where (Π', H', Λ') is any solution of (4.68) implying that $\mathcal{C}^*(\Sigma) \subset \ker \Pi'_{\max}$, especially $\mathcal{C}^*(\Sigma) \subset \ker \Pi'_{\max}$. According to the definition of $\mathcal{C}^*(\Sigma)$ there exists a finite input sequence producing zero output and directing the origin to the vector α . Denoting by $\eta_0, \eta_1, \dots, \eta_j$ this sequence of inputs and by $\xi_0, \xi_1, \dots, \xi_{j-1}$ the sequence of state vectors produced by using this input sequence the following system of equations holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi_0 & \xi_1 & \dots & \xi_{j-1} & 0 \\ \eta_0 & \eta_1 & \dots & \eta_{j-1} & \eta_j \end{bmatrix} = \begin{bmatrix} \alpha & \xi_0 & \dots & \xi_{j-1} \\ 0 & 0 & \dots & 0 \end{bmatrix} .$$

Multiplying this equation by $[\Pi', H']$ from the left and using (4.68) we obtain that

$$[\Pi'\alpha, \Pi'\xi_0, \dots, \Pi'\xi_{j-1}] = [\Lambda'\Pi'\xi_0, \Lambda'\Pi'\xi_1, \dots, \Lambda'\Pi'\xi_{j-1}, 0] .$$

Consequently,

$$\Pi'\xi_{j-1} = 0, \dots, \Pi'\xi_0 = 0, \Pi'\alpha = 0 .$$

I.e. $\mathcal{C}^*(\Sigma) \subset \ker \Pi'$, concluding the proof of the theorem. ■

Similar proof gives the following statement:

$$(\mathcal{C}_{\text{left}}^*)^\perp(\Sigma) = \mathcal{V}^*(\Sigma) ,$$

implying the following corollary:

Corollary 4.1. *Assume that the realization*

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

is minimal. Then

$$\dim Z(F) = \dim Z_{\text{left}}(F) , \tag{4.69}$$

$$\dim \mathcal{W}(\ker F) = \dim \mathcal{W}(\text{Im}_{\text{left}} F) , \tag{4.70}$$

$$\dim Z_\infty(F) = \dim Z_{\infty, \text{left}}(F) , \tag{4.71}$$

$$\dim \mathcal{W}(\text{Im} F) = \dim \mathcal{W}(\ker_{\text{left}} F) . \tag{4.72}$$

The following theorem shows that there is a deeper connection between the left and right finite zero spaces of F .

Theorem 4.2. *Assume that the realization of F given by*

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

is minimal. Then the left and right finite zero matrices Λ_f , $\Lambda_{f,\text{left}}$ are similar.

PROOF. Consider maximal solutions of equations (3.15), (3.21) and (4.68) and the corresponding “left” version of (3.21).

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi_{fzk} \\ H_{fzk} \end{bmatrix} = \begin{bmatrix} \Pi_{fzk} \Lambda_{fzk} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} B \\ D \end{bmatrix} R_0 = \begin{bmatrix} \Pi_{fzk} \alpha_0 \\ 0 \end{bmatrix}, \quad (4.73)$$

$$[\Pi'_{fzk}, H'_{fzk}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\Lambda'_{fzk} \Pi'_{fzk}, 0], \quad R'_0 [C, D] = [\alpha'_0 \Pi'_{fzk}, 0], \quad (4.74)$$

where for the sake of simplicity the “left” is indicated by the notation \cdot'

Without loss of generality we might assume that these matrices are partitioned as it is described in Remark 3.9. (Applying it also to the “left” structure, as well.):

$$\Lambda_{fzk} = \begin{bmatrix} \Lambda_k & \Lambda_{kf} \\ 0 & \Lambda_f \end{bmatrix} \alpha_0 = \begin{bmatrix} \alpha_k \\ 0 \end{bmatrix} \quad (4.75)$$

$$\Lambda'_{fzk} = \begin{bmatrix} \Lambda'_k & 0 \\ \Lambda'_{kf} & \Lambda'_f \end{bmatrix}, \quad \alpha'_0 = [\alpha'_k, 0], \quad (4.76)$$

where the pair (Λ_k, α_k) is controllable, (α'_k, Λ'_k) is observable.

Partitioning the matrices $\Pi_{fzk}, H_{fzk}, \Pi'_{fzk}, H'_{fzk}$ accordingly, we get that

$$\text{Im} \Pi_k = \mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma), \quad \text{Im} \Pi_{fzk} = \mathcal{V}^*(\Sigma), \quad (4.77)$$

$$\text{Im}_{\text{left}} \Pi'_k = \mathcal{C}^*_{\text{left}}(\Sigma) \cap \mathcal{V}^*_{\text{left}}(\Sigma), \quad \text{Im}_{\text{left}} \Pi'_{fzk} = \mathcal{V}^*_{\text{left}}(\Sigma), \quad (4.78)$$

and Theorem 4.1 implies that

$$\Pi'_{fzk} \Pi_k = 0, \quad \Pi'_k \Pi_{fzk} = 0$$

Multiplying the first equation in (4.73) from the left by $[\Pi'_{fzk}, H'_{fzk}]$ and using the first equation in (4.74) we obtain that

$$\begin{bmatrix} \Pi'_k \\ \Pi'_f \end{bmatrix} [\Pi_k, \Pi_f] \begin{bmatrix} \Lambda_k & \Lambda_{kf} \\ 0 & \Lambda_f \end{bmatrix} = \begin{bmatrix} \Lambda'_k & 0 \\ \Lambda'_{kf} & \Lambda'_f \end{bmatrix} \begin{bmatrix} \Pi'_k \\ \Pi'_f \end{bmatrix} [\Pi_k, \Pi_f].$$

Shortly

$$\Pi'_f \Pi_f \Lambda_f = \Lambda'_f \Pi'_f \Pi_f.$$

Since Theorem 4.1 gives also that the matrix $\Pi'_f \Pi_f$ is square and nonsingular the similarity of the matrix Λ_f and Λ'_f is obtained. In fact

$$(\Pi'_f \Pi_f) \Lambda_f (\Pi'_f \Pi_f)^{-1} = \Lambda'_f .$$

In other words, the finite left and right zero matrices of the function F are similar to each other. ■

4.1 Connection between the values of F and K_0 at a given point $\lambda \in \mathbb{C}$

Assume that the matrices (A, B, C, D) provide a *minimal* realization of F and consider the function K_0 given in Remark 3.8 (or K_β defined in Remark 3.10) “generating” the kernel of F (in the sense that for any q -tuple g of rational functions for which $Fg \equiv 0$ holds there exists a (vector-valued) rational function h such that $g = K_0 h$. The converse statement obviously holds)

Since $FK_0 = 0$, if both functions F and K_0 are analytic at a given $\lambda' \in \mathbb{C}$, the same connection holds for the values of these functions taken at λ' . I.e.

$$F(\lambda') K_0(\lambda') = 0 .$$

In other words the row-vectors of $F(\lambda')$ are orthogonal to the column-vectors of $K_0(\lambda')$.

More generally, consider a solution of the set of equations:

$$[Y' , Z'] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\lambda' Y' , h'] \quad (4.79)$$

(where now λ' can also be a matrix) implying obviously that

$$(zI - \lambda')^{-1} (Z' F(z) - h') = -Y' (zI - A)^{-1} B .$$

Thus, if the spectra of A and λ' are disjoint then h' determines the “directional” values and derivatives of F taken at the eigenvalues of λ' .

Now multiplying from the right by $\begin{bmatrix} \Pi_k \\ H_k \end{bmatrix}$ and by $\begin{bmatrix} 0 \\ R_0 \end{bmatrix}$ we obtain the following equations

$$\begin{aligned} \lambda' Y' \Pi_k + h' H_k &= Y' \Pi_k \Lambda_k \\ h' R_0 &= Y' \Pi_k \alpha_k . \end{aligned}$$

In other words

$$[Y' \Pi_k , -h'] \begin{bmatrix} \Lambda_k & \alpha_0 \\ H_k & R_0 \end{bmatrix} = [\lambda' Y' \Pi_k , 0] , \quad (4.80)$$

thus

$$(zI - \lambda')^{-1} h' (R_0 + H_k (zI - \Lambda_k)^{-1} \alpha_k) = Y' \Pi_k (zI - \Lambda_k)^{-1} \alpha_k .$$

Shortly

$$(zI - \lambda')^{-1} h' K_0(z) = Y' \Pi_k (zI - \Lambda_k)^{-1} \alpha_k .$$

Now, if the spectra of λ' and Λ_k are disjoint then the pair (λ', h') is a right-zero pair of K_0 .

Summarizing these considerations: if the spectra of λ' and that of A and Λ_k are disjoint then the assumption $(zI - \lambda')^{-1} (Z' F(z) - h')$ is analytic on the set of eigenvalues of λ' implies that $(zI - \lambda')^{-1} h' K_0(z)$ is analytic there.

In that special case, when λ' is a matrix in Jordan-form, then equations (4.79) and (4.80) establish connections between the “directional” derivatives of F and K_0 taken at the eigenvalues of λ' .

The following theorem shows that the converse statement also holds true. Under some conditions, if a pair is a right zero pair of the function K_0 , then the same pair determines also interpolation values of the function F , i.e. at the same locations using appropriately defined directions the directional values of F coincide with the zero directions of K_0 .

Theorem 4.3. *Assume that the realization of F given by*

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

is minimal. Consider maximal solutions $(\Pi_{max}, H_{max}, \Lambda_{max})$ of (3.10) and (R_0, α_0) of (3.21) for which $\ker \Pi_{max} = \{0\}$. Define the function K_0 according to Remark 3.8.

Assume that for some matrices λ' and h' the product

$$(zI - \lambda')^{-1} h' K_0(z) \quad \text{is analytic on the spectrum of } \lambda' .$$

If the spectrum of λ' is disjoint from that of Λ_f and A , then there exists a matrix Z' such that the product

$$(zI - \lambda')^{-1} (Z' F(z) - h') \quad \text{is analytic on the spectrum of } \lambda' .$$

PROOF. The proof of the theorem is based on the following lemma which is valid under more general assumptions, as well.

Lemma 4.1. *Assume that the realization of F given by*

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

is minimal. Consider maximal solutions $(\Pi_{max}, H_{max}, \Lambda_{max})$ of (3.10) and (R_0, α_0) of (3.21) for which $\ker \Pi_{max} = \{0\}$. (Let us recall that under the minimality assumption the subscripts .max and .fzk mean the same.) Assume that the matrices σ', h', λ' provide a solution of the equations

$$[\sigma' , -h'] \begin{bmatrix} \Lambda_{max} & \alpha_0 \\ H_{max} & R_0 \end{bmatrix} = [\lambda' \sigma' , 0] , \quad (4.81)$$

Then there exist matrices Y', Z' such that $Y' \Pi_{max} = \sigma'$ and equations (4.79) hold.

PROOF. Before proving the lemma let us observe that multiplying the equation (4.79) from the right by $\begin{bmatrix} \Pi_{fzk} \\ H_{fzk} \end{bmatrix}$ and by $\begin{bmatrix} 0 \\ R_0 \end{bmatrix}$ we obtain the following equations

$$\begin{aligned} \lambda' Y' \Pi_{fzk} + h' H_{fzk} &= Y' \Pi_{fzk} \Lambda_{fzk} \\ h' R_0 &= Y' \Pi_{fzk} \alpha_0 . \end{aligned}$$

In other words

$$[Y' \Pi_{fzk} , -h'] \begin{bmatrix} \Lambda_{fzk} & \alpha_0 \\ H_{fzk} & R_0 \end{bmatrix} = [\lambda' Y' \Pi_{fzk} , 0] . \quad (4.82)$$

Thus the present lemma essentially states – using the assumption that the realization of F is minimal – that equations (4.79) and (4.81) are equivalent.

For proving the lemma first notice that since according to our assumption $\ker(\Pi_{\max}) = \{0\}$ there exists a matrix Y'_1 such that $Y'_1 \Pi_{\max} = \sigma'$. Then $\sigma' \alpha_0 = Y'_1 \Pi_{\max} \alpha_0 = Y'_1 B R_0$. Thus

$$(Y'_1 B - h') R_0 = 0$$

The maximality of the solution of equation (3.21) gives that $\ker \begin{bmatrix} B & \Pi_{\max} \\ D & 0 \end{bmatrix} = \text{Im} \begin{bmatrix} R_0 \\ 0 \end{bmatrix}$ thus any row vector orthogonal to the columns of R_0 can be written in the form $\eta B + \xi D$, where $\eta \Pi_{fzk} = 0$. Thus there exist matrices Y'_2, Z'_1 such that

$$Y'_1 B - h' = Y'_2 B + Z'_1 D , \quad Y'_2 \Pi_{fzk} = 0 \quad (4.83)$$

Also the first equation in (4.82) gives that

$$Y'_1 \Pi_{fzk} \Lambda_{fzk} - h' H_{fzk} - \lambda' Y'_1 \Pi_{fzk} = 0 .$$

Expressing $\Pi_{fzk} \Lambda_{fzk} = A \Pi_{fzk} + B H_{fzk}$ and h' from (4.83) we get that

$$Y'_1 A \Pi_{fzk} + Y'_1 B H_{fzk} - Y'_1 B H_{fzk} + Y'_2 B H_{fzk} + Z'_1 D H_{fzk} - \lambda' Y'_1 \Pi_{fzk} = 0 .$$

Thus

$$Y'_1 A \Pi_{fzk} - \lambda' Y'_1 \Pi_{fzk} + Y'_2 \Pi_{fzk} \Lambda_{fzk} - Y'_2 A \Pi_{fzk} - Z'_1 C \Pi_{fzk} = 0 .$$

Using the identity $Y'_2 \Pi_{fzk} = 0$ we arrive at the following equation

$$[(Y'_1 - Y'_2) A - Z'_1 C - \lambda' (Y'_1 - Y'_2)] \Pi_{fzk} = 0 \quad (4.84)$$

Now Theorem 4.1 implies that the row vectors orthogonal to $\text{Im}(\Pi_{fzk})$ are in $\mathcal{C}_{\text{left}}^*(\Sigma)$. I.e. there exists an integer l and sequences of matrices $\xi_0 = 0, \xi_1, \dots, \xi_l$ and $\mu_0, \mu_1, \dots, \mu_{l-1}$ such that

$$[\xi_j , \mu_j] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\xi_{j+1} , 0] , \quad j = 0, 1, \dots, l-1 . \quad (4.85)$$

and

$$\xi_l = (Y'_1 - Y'_2) A - Z'_1 C - \lambda' (Y'_1 - Y'_2) .$$

This latter equation together with (4.83) can be written as follows

$$[(Y'_1 - Y'_2) , -Z'_1] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\xi_l + \lambda' (Y'_1 - Y'_2) , h'] \quad (4.86)$$

Multiplying (4.85) from the left by $(\lambda')^{l-1-j}$ taking the sum from $j = 0$ to $l-1$ and subtracting it from (4.86) we obtain that

$$\begin{aligned} \left[\left(Y'_1 - Y'_2 - \sum_{j=0}^{l-1} (\lambda')^{l-1-j} \xi_j \right) , -Z'_1 - \sum_{j=0}^{l-1} (\lambda')^{l-1-j} \mu_j \right] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ = \left[\xi_l + \lambda' (Y'_1 - Y'_2) - \sum_{j=0}^{l-1} (\lambda')^{l-1-j} \xi_{j+1} , h' \right] \end{aligned} \quad (4.87)$$

Introducing the notation (using that $\xi_0 = 0$)

$$\begin{aligned} Y' &= Y'_1 - Y'_2 - \sum_{j=1}^{l-1} (\lambda')^{l-1-j} \xi_j \\ Z' &= -Z'_1 - \sum_{j=0}^{l-1} (\lambda')^{l-1-j} \mu_j \end{aligned}$$

we get that

$$[Y' , Z'] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\lambda' Y' , h']$$

concluding the proof of the lemma. ■

Let us return to the proof of the theorem. We might assume w.l.o.g. that the matrices are partitioned according to (3.29) and (3.30). Then the function K_0 has the *minimal* realization $K_0 = R_0 + H_k (zI - \Lambda_k)^{-1} \alpha_k$. According to part (i) of Theorem 3.1, if for the pair (λ', h') the product $(zI - \lambda')^{-1} h' K_0(z)$ is analytic at the eigenvalues of λ' then there exists a solution σ' of the equation

$$[\sigma' , -h'] \begin{bmatrix} \Lambda_k & \alpha_k \\ H_k & R_0 \end{bmatrix} = [\lambda' \sigma' , 0] .$$

Using the assumption that the spectra of λ' and Λ_f are disjoint we get that the Sylvester-equation

$$\lambda' \sigma'' - \sigma'' \Lambda_f = \sigma' \Lambda_{kf} - h' H_f$$

has also a solution in σ'' . In other words the equation

$$[\sigma' , \sigma'' , -h'] \begin{bmatrix} \Lambda_k & \Lambda_{kf} & \alpha_k \\ 0 & \Lambda_f & 0 \\ H_k & H_f & R_0 \end{bmatrix} = [\lambda' \sigma' , \lambda' \sigma'' , 0]$$

holds. Applying Lemma 4.1 we obtain that there exists matrices Y', Z' such that $Y' \Lambda_{\max} = [\sigma' , \sigma'']$ and equation (4.79) holds.

Invoking now part (ii) of Theorem 3.1 (or directly computing the product) – using that λ' and A have no common eigenvalues we obtain that

$$(zI - \lambda')^{-1} (Z'F(z) - h') = -Y' (zI - A)^{-1} B$$

is analytic on the spectra of λ' , concluding the proof of the theorem. \blacksquare

REMARK 4.19 Let us note that in the previous theorem instead of K_0 any other function K_β can be used, due to the fact that the matrices $(\Pi_{fzk}, H_{fzk} + \alpha_0\beta, \Lambda_{fzk} + R_0\beta)$ are also maximal solutions of the equation (3.10) due to the assumption that a minimal realization of F was considered.

Corollary 4.2. *Consider a complex number $\lambda \in \mathbb{C}$ which is not a finite zero of F , and assume that the functions F and K_0 are analytic at λ . Choosing $\lambda' = \lambda I$ (where I has appropriate size) the previous theorem gives that the row-space spanned by the row-vectors of $F(\lambda)$ generate the orthogonal complement of the column space generated by the column-vectors of $K_0(\lambda)$.*

4.2 Further elimination via factorization: $\mathcal{W}(\ker_{\text{left}} F)$

In Section 3.1.5 a special factorization of function F of the form $F = F_{\mathbf{r}} L_\beta^*$ was discussed, where the inner function L_β was constructed via the square inner extension of the function K_β . (This latter one generates the module $\mathcal{W}(\ker F)$. See Theorems 3.5 and 3.6.)

Applying the same idea we can eliminate the left kernel module of F , as well. But in order to eliminate both the left and right kernel modules of F at the same time we have to consider the left kernel-module of $F_{\mathbf{r}} = F L_\beta^*$. To this aim first we have to consider maximal solution of the "left" version of equation (3.10) for the realization (3.44) of $F_{\mathbf{r}}$ given in Theorem 3.7. As we have seen earlier, this maximal solution is connected to the subspace $\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}})$. Theorem 3.7 provides explicit connections between the various subspaces used in geometric control theory (maximal output-nulling controlled invariant subspace, minimal input-containing subspace, maximal output-nulling reachability subspace) determined by the given realizations of the functions F and $F_{\mathbf{r}}$, especially showing that while $\mathcal{V}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}^*(\Sigma)$, the minimal input-containing subspace reduces, $\mathcal{C}^*(\Sigma_{\mathbf{r}}) \subset \mathcal{C}^*(\Sigma)$, in such a way that the intersection $\mathcal{R}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}^*(\Sigma_{\mathbf{r}})$ becomes trivial. As a consequence of this – using Theorem 4.1 – $\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}})$ becomes larger than $\mathcal{V}_{\text{left}}^*(\Sigma)$.

Theorem 4.4 provides a detailed picture of this question in terms of state-space matrices solving the "left" version of equation (3.10) for the realization (3.44) of $F_{\mathbf{r}}$ given in Theorem 3.7.

To formulate this theorem we first need an auxiliary statement formulated as a corollary of the following version of Lemma 4.1.

Lemma 4.2. *Assume that the realization of F given by*

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

is minimal. Consider maximal solutions $(\Pi_{max}, H_{max}, \Lambda_{max})$ of (3.10) and (R_0, α_0) of (3.21) for which $\ker \Pi_{max} = \{0\}$ assuming that the column vectors of R_0 are orthonormal and the matrices are partitioned according to (3.29) and (3.30).

Assume that the matrices σ', h', λ' provide a solution of the equations

$$[\sigma' , -h'] \begin{bmatrix} \Lambda_k & \alpha_k \\ H_k & R_0 \end{bmatrix} = [\lambda' \sigma' , 0] , \quad (4.88)$$

Then there exist matrices Y', Z', V' such that $Y' \Pi_k = \sigma'$, the row vectors of V' are in $\mathcal{V}_{\text{left}}^*(\Sigma)$, and equations

$$[Y' , Z'] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\lambda' Y' + V' , h'] \quad (4.89)$$

hold.

PROOF. Let us make the obvious changes in the proof of Lemma 4.1, i.e. instead of considering Λ_{max}, \dots use Λ_k, \dots .

Since according to our assumption $\ker(\Pi_k) = \{0\}$ there exists a matrix Y'_1 such that $Y'_1 \Pi_k = \sigma'$.

Following the steps in the previous proof we – instead of equation (4.84) – arrive at the equation

$$[(Y'_1 - Y'_2) A - Z'_1 C - \lambda' (Y'_1 - Y'_2)] \Pi_k = 0 ,$$

where – as before – $Y'_2 \Pi_{max} = 0$, i.e. the rows of Y'_2 are in $\mathcal{C}_{\text{left}}^*(\Sigma)$.

Now according to Theorem 4.1 and its immediate consequence the row vectors orthogonal to $\text{Im}(\Pi_k) = \mathcal{V}^*(\Sigma) \cap \mathcal{C}^*(\Sigma)$ are in $\mathcal{C}_{\text{left}}^*(\Sigma) \vee \mathcal{V}_{\text{left}}^*(\Sigma)$ we have that

$$((Y'_1 - Y'_2) A - Z'_1 C - \lambda' (Y'_1 - Y'_2)) = \xi_l + V'$$

for some matrices ξ_l and V' where the rows of ξ_l are in $\mathcal{C}_{\text{left}}^*(\Sigma)$, while those of V' are in $\mathcal{V}_{\text{left}}^*(\Sigma)$.

Thus instead of (4.86) we obtain that

$$[(Y'_1 - Y'_2) , -Z'_1] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\xi_l + \lambda' (Y'_1 - Y'_2) + V' , h'] .$$

Embedding the rows of ξ_l into sequences in $\mathcal{C}_{\text{left}}^*(\Sigma)$ and continuing the proof as it was done in Lemma 4.1 we get that equation

$$[Y' , Z'] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\lambda' Y' + V' , h']$$

holds, concluding the proof of the present lemma. ■

REMARK 4.20 Let us point out that the matrices Y', Z' and V' can be chosen in such a way that for the matrix V' the following more stringent condition holds: considering any (maximal) complementary subspace of $\mathcal{R}_{\text{left}}^*(\Sigma)$ in $\mathcal{V}_{\text{left}}^*(\Sigma)$ the row vectors of V' are in this subspace.

Let us observe that the equations (3.34) and (3.35) (or equivalently the Riccati-equation (3.38) and (3.37)) can be written as

$$[\sigma, (H_k + R_0\beta_k)^*] \begin{bmatrix} \Lambda_k + \alpha_k\beta_k & \alpha_k \\ H_k + R_0\beta_k & R_0 \end{bmatrix} = [-(\Lambda_k + \alpha_k\beta_k)^* \sigma, 0].$$

Thus Lemma 4.2 can be applied giving the following corollary.

Corollary 4.3. *Consider a **minimal realization** of F given as*

$$F(z) \sim \left(\frac{A}{C} \middle| \frac{B}{D} \right)$$

Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15). Consider a maximal solution – in terms of α_0 and R_0 – of the equation (3.21) assuming – w.l.o.g. – that the column-vectors of the matrix R_0 are orthonormal and the matrices are partitioned according to (3.29) and (3.30).

Denote by σ the positive definite solution of the Riccati-equation (3.38). Set

$$\beta_k^* = -H_k^* R_0 - \sigma \alpha_k.$$

then there exists matrices Y_k, Z_k and V_k such that the rows of V_k are in $\mathcal{V}_{\text{left}}^(\Sigma)$*

$$Y_k \Pi_k = \sigma, \tag{4.90}$$

and

$$[Y_k, Z_k] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [-(\Lambda_k + \alpha_k\beta_k)^* Y_k + V_k, -(H_k + R_0\beta_k)^*]. \tag{4.91}$$

After these preliminary statements we can formulate the theorem determining the subspace $\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}})$ for the given realization of $F_{\mathbf{r}}$.

Theorem 4.4. *Consider a **minimal realization** of F given by*

$$F(z) \sim \left(\frac{A}{C} \middle| \frac{B}{D} \right)$$

Assume that the columns of the function $H_{fzk}(zI - \Lambda_{fzk})^{-1}$ provide a basis in $Z(F) \oplus \mathcal{W}(\ker F)$. Let Π_{fzk} be the corresponding solution of (3.15). Consider a maximal solution – in terms of α_0 and R_0 – of the equation (3.21) assuming – w.l.o.g. – that the column-vectors of the matrix R_0 are orthonormal and the matrices are partitioned according to (3.29) and (3.30).

Consider the function $F_{\mathbf{r}}$ determined in Theorem 3.7 with the realization given in (3.54). Assume that Π'_{fzk} , Λ'_{fzk} and H'_{fzk} define a maximal solution of

$$[\Pi'_{fzk}, H'_{fzk}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\Lambda'_{fzk} \Pi'_{fzk}, 0],$$

assuming that the left kernel of Π'_{fzk} is trivial.

Then the maximal solution of the equation

$$[\bar{\Pi}', \bar{H}'] \begin{bmatrix} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ C & DL_0 \end{bmatrix} = [\bar{\Lambda}' \bar{\Pi}', 0] , \quad (4.92)$$

is provided by

$$\bar{\Pi}' = \begin{bmatrix} \Pi'_{fzk} \\ Y_k \end{bmatrix} , \quad \bar{H}' = \begin{bmatrix} H'_{fzk} \\ Z_k \end{bmatrix} , \quad \bar{\Lambda}' = \begin{bmatrix} \Lambda'_{fzk} & 0 \\ \Delta' & -(\Lambda_k + \alpha_k \beta_k)^* \end{bmatrix} , \quad (4.93)$$

where the matrices Y_k, Z_k are given in Corollary 4.3 and Δ' is defined as the unique solution of equation

$$\Delta' \Pi'_{fzk} = V_k .$$

PROOF. Equations $Y_k \Pi_k = \sigma$, $R_0^* L_0 = 0$ and Corollary 4.3 imply that the matrices defined in (4.93) satisfy equation (4.92).

To prove that it gives amximal solution first let us determine the rank of the maximal solution. For a maximal solution we have that

$$\begin{aligned} \text{rank}(\bar{\Pi}') &= \dim \mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) \\ &= n - \dim \mathcal{C}^*(\Sigma_{\mathbf{r}}) \\ &= n - (\dim \mathcal{C}^*(\Sigma) - \text{rank}(\Pi_k)) \\ &= \text{rank}(\Pi'_{fzk}) + \text{rank}(\Pi_k) , \end{aligned}$$

where n is the dimension of the state space.

Equation $Y_k \Pi_k = \sigma > 0$ gives that $\text{rank}(Y_k) = \text{rank}(\sigma) = \text{rank}(\Pi_k)$. Since $\Pi'_{fzk} \Pi_k = 0$, while $Y_k \Pi_k$ is positive definite, the left kernel of the matrix $\begin{bmatrix} \Pi'_{fzk} \\ Y_k \end{bmatrix}$ is trivial and its rank equals to the rank of the maximal solution, concluding thus the proof of the theorem. ■

Now let us return to the both sided factorization of F . Let us apply the factorization ideas given in Section 3.1.5 for eliminating the left kernel of F or equivalently of $F_{\mathbf{r}}$. Since according to Theorem 3.7

$$\mathcal{V}^*(\Sigma_{\mathbf{r}}) \vee \mathcal{C}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}^*(\Sigma) \vee \mathcal{C}^*(\Sigma)$$

Theorem 4.1 implies that

$$\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}}) = \mathcal{V}_{\text{left}}^*(\Sigma) \cap \mathcal{C}_{\text{left}}^*(\Sigma)$$

giving that F and $F_{\mathbf{r}}$ determine that same “left-kernel” flat inner function, denoted by $K'_{\beta'}$. (I.e. $K'_{\beta'} K'_{\beta'}^* = I$.)

With obvious notation:

$$K'_{\beta'}(z) = R'_0 + \alpha'_k (zI - (\Lambda'_k + \beta'_k \alpha'_k))^{-1} (H'_k + \beta'_k R'_0) ,$$

where

$$\beta'_k = -\sigma' \alpha'_k{}^* - H'_k R'_0{}^* \quad (4.94)$$

and σ' is the positive definite solution of the Riccati-equation

$$\left(\Lambda'_k - H'_k R'_0{}^* \alpha'_k \right) \sigma' + \sigma' \left(\Lambda'_k - H'_k R'_0{}^* \alpha'_k \right)^* - \sigma' \alpha'_k{}^* \alpha'_k \sigma' + H'_k \left(I - R'_0{}^* R'_0 \right) H'_k{}^* = 0 . \quad (4.95)$$

Consider the square inner extension of $K'_{\beta'}$:

$$K'_{\beta', \text{ext}} = \begin{bmatrix} K'_{\beta'} \\ L'_{\beta'} \end{bmatrix} , \quad (4.96)$$

where

$$L'_{\beta'} = L'_0 - L'_0 (H'_k + \beta'_k R'_0{}^*)^* (zI - (\Lambda'_k + \beta'_k \alpha'_k))^{-1} (H'_k + \beta'_k R'_0{}^*)$$

and $\begin{bmatrix} R'_0 \\ L'_0 \end{bmatrix}$ is a unitary matrix.

Theorem 3.7 applied to the left zero structure gives the factorization summarized in the following theorem.

Theorem 4.5. *Let a minimal realization of F be given by*

$$F(z) \sim \left(\frac{A}{C} \middle| \frac{B}{D} \right) .$$

Consider maximal solutions of the equations (4.73) and (4.74) assuming that the columns (rows) of R_0 (R'_0) are orthonormal and they are partitioned as it is described in Remark 3.9 (applying it also to the “left” structure, as well). Denote by σ (σ') the solutions of the Riccati-equations (3.38) ((4.95) respectively). Define the functions L_β and $L'_{\beta'}$ by (3.42) and (4.96).

Consider the function $F_{\mathbf{r}\mathbf{l}} = L'_{\beta'} F L_\beta$. Then

(i) the function F has the following factorization

$$F = L_{\beta'}^* F_{\mathbf{r}\mathbf{l}} L_\beta^* ,$$

where $F_{\mathbf{r}\mathbf{l}}$ has the realization

$$F_{\mathbf{r}\mathbf{l}}(z) \sim \left(\frac{A}{L'_0 (C + H'_k{}^* \sigma'^{-1} \Pi'_k)} \middle| \frac{(B + \Pi_k \sigma^{-1} H_k^*) L_0}{L'_0 D L_0} \right) \quad (4.97)$$

(ii) if

(a) if all the eigenvalues of A have non-positive real part , or

(b) the matrices

A and $-A^*$ have no common eigenvalues, and
the pair (A, \overline{C}^*) is stabilizable (in continuous time sense)
the pair (\overline{B}^*, A) is detectable (in continuous time sense),

where $\overline{C} = CP + DB^*$ and P is the solution of the Lyapunov-equation

$$AP + PA^* + BB^* = 0, \quad (4.98)$$

$\overline{B} = QB + C^*D$ and Q is the solution of the Lyapunov-equation

$$QA + A^*A + C^*C = 0 \quad (4.99)$$

then the realization (4.97) above of $F_{\mathbf{r}}$ is minimal.

(iii) if the realization given in (4.97) of $F_{\mathbf{r}}$ is minimal then for the maximal output-nulling controlled invariant subspace $\mathcal{V}^*(\Sigma_{\mathbf{r1}})$ and for the minimal input-containing subspace $\mathcal{C}^*(\Sigma_{\mathbf{r1}})$ of the realization of $F_{\mathbf{r1}}$ the following identities hold:

$$\mathcal{C}^*(\Sigma_{\mathbf{r1}}) \cap \mathcal{V}^*(\Sigma_{\mathbf{r1}}) = \{0\} \quad (4.100)$$

$$\mathcal{C}^*(\Sigma_{\mathbf{r1}}) \vee \mathcal{V}^*(\Sigma_{\mathbf{r1}}) = \mathbb{C}^n \quad (4.101)$$

$$\mathcal{C}^*(\Sigma_{\mathbf{r1}}) = \mathcal{C}^*(\Sigma_{\mathbf{r}}) \quad (4.102)$$

$$(\mathcal{C}^*(\Sigma) \cap \mathcal{V}^*(\Sigma)) \vee \mathcal{C}^*(\Sigma_{\mathbf{r1}}) = \mathcal{C}^*(\Sigma) \quad (4.103)$$

$$\mathcal{V}^*(\Sigma_{\mathbf{r1}}) \cap (\mathcal{C}^*(\Sigma) \vee \mathcal{V}^*(\Sigma)) = \mathcal{V}^*(\Sigma) \quad (4.104)$$

and

$$\mathcal{W}(\ker F_{\mathbf{r1}}) = \{0\}, \quad \mathcal{W}(\text{Im} F_{\mathbf{r1}}) = \{0\},$$

and the function $F_{\mathbf{r1}}$ is invertible;

(iv) if the realization (4.97) is minimal then the finite zero matrix of $F_{\mathbf{r1}}$ is given by

$$\begin{bmatrix} -\sigma^{-1}(\Lambda_k + \alpha_k \beta_k)^* \sigma, & \Lambda_{kf} + \sigma^{-1}(H_k + R_0 \beta_k)^* H_f, & \sigma^{-1}(H_k + R_0 \beta_k)^* Z'_k \\ 0 & \Lambda_f & \Delta \\ 0 & 0 & -(\Lambda'_k + \beta'_k \alpha'_k)^* \end{bmatrix} \quad (4.105)$$

for some matrices Z'_k and Δ .

Note that the eigenvalues of Λ_f determine the finite zeros of the function F , while the matrices Λ_k, α_k and Λ'_k, α'_k are connected to the right and left kernel spaces $-\mathcal{W}(\ker F)$, $\mathcal{W}(\ker_{\text{left}} F) - \text{of } F$, respectively.

PROOF. Substitute into $(L'_{\beta'})^* F_{\mathbf{r1}} L_{\beta}^*$ the definition of $F_{\mathbf{r1}}$:

$$L_{\beta'}^* F_{\mathbf{r1}} L_{\beta}^* = L_{\beta'}^* L'_{\beta'} F L_{\beta} L_{\beta}^* = (L_{\beta'}^* L'_{\beta'} + K_{\beta'}^* K'_{\beta'}) F (L_{\beta} L_{\beta}^* + K_{\beta} K_{\beta}^*) = F$$

Straightforward computation (or immediate application of Theorem 3.7 gives that the function $F_{\mathbf{r}1}$ has the realization:

$$\begin{aligned}
F_{\mathbf{r}1}(z) &= L'_{\beta'} F_{\mathbf{r}} \\
&= \left(L'_0 - L'_0 H_k'^* \sigma'^{-1} \left(zI - (\Lambda'_k + \beta'_k \alpha'_k) \right) \right)^{-1} \left(H'_k + \beta'_k R'_0 \right) \\
&\quad \left(DL_0 + C (zI - A)^{-1} (B + \Pi_k \sigma^{-1} H_k^*) L_0 \right) \\
&= L'_0 DL_0 + L'_0 \left(C + H_k'^* \sigma'^{-1} \Pi'_k \right) (zI - A)^{-1} (B + \Pi_k \sigma^{-1} H_k^*) L_0 ,
\end{aligned}$$

using the identities

$$\begin{aligned}
\left(H'_k + \beta'_k R'_0 \right) C &= \Pi'_k (zI - A) - (zI - (\Lambda'_k + \beta'_k \alpha'_k)) \Pi'_k \\
\left(H'_k + \beta'_k R'_0 \right) DL_0 &= -\Pi'_k (B + \Pi_k \sigma^{-1} H_k^*) L_0 ,
\end{aligned}$$

proving (i).

(ii) Both a) and b) parts can be proven using part (iii) Theorem 3.7. In fact, under the condition that all eigenvalues of A have non-positive real part part (iii) a) of Theorem 3.7 gives that the reachability subspaces $\langle A \mid B \rangle$ and $\langle A \mid (B + \Pi_k \sigma^{-1} H_k^*) L_0 \rangle$ coincide. Applying the "left" version of this result we obtain that the non-observability subspaces of the pairs (C, A) and $(L'_0(C + H_k'^* \sigma'^{-1} \Pi'_k), A)$ coincide. But according to our assumption the realization F is minimal, consequently the realization $F_{\mathbf{r}1}$ above is also minimal.

Concerning the b) part of this statement now part (iii) b) of Theorem and its "left" version gives again that reachability subspaces above and non-observability subspaces above coincide, giving again the minimality of the realization (4.97).

(iii) Denote by $\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}1})$, $\mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}1})$ the maximal output-nulling controlled invariant subspace and the minimal input-containing subspace of the realization of $F_{\mathbf{r}1}$ given in (4.97) with respect to the left multiplication.

Then Theorem 4.1 allows us to transform the results of Theorem 3.7 to the left zero structure of $F_{\mathbf{r}}$. Consequently,

$$\begin{aligned}
\mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}}) &= \mathcal{C}_{\text{left}}^*(\Sigma) , \\
\mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}}) \vee \mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) &= \mathbb{C}^n , \\
(\mathcal{C}_{\text{left}}^*(\Sigma) \vee \mathcal{V}_{\text{left}}^*(\Sigma)) \cap \mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) &= \mathcal{V}_{\text{left}}^*(\Sigma) .
\end{aligned}$$

The pair $(A, (B + \Pi_k \sigma^{-1} H_k^*) L_0)$ is reachable due to the our assumption that the realization (4.97) is minimal. Consequently, Theorem 3.7 can be applied to the "left" factorization of $F_{\mathbf{r}}$ yielding that

$$\begin{aligned}
\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}1}) &= \mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) , \\
\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}1}) \cap \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}1}) &= \{0\} , \\
(\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}}) \cap \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}})) \vee \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}1}) &= \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}}) .
\end{aligned}$$

In the last equation taking on both sides the generated subspace by $\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}\mathbf{l}}) = \mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}})$ we obtain that

$$\mathcal{V}_{\text{left}}^*(\Sigma_{\mathbf{r}\mathbf{l}}) \vee \mathcal{C}_{\text{left}}^*(\Sigma_{\mathbf{r}\mathbf{l}}) = \mathbb{C}^n .$$

Invoking Theorem 4.1 (i.e. taking the orthogonal complements of these subspaces) we obtain that

$$\begin{aligned} \mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}}) &= \mathcal{C}^*(\Sigma_{\mathbf{r}}) , \\ \mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}}) \vee \mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}}) &= \mathbb{C}^n , \\ \mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}}) \cap \mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}}) &= \{0\} \\ (\mathcal{C}^*(\Sigma_{\mathbf{r}}) \vee \mathcal{V}^*(\Sigma_{\mathbf{r}})) \cap \mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}}) &= \mathcal{V}^*(\Sigma_{\mathbf{r}}) . \end{aligned}$$

The last equation can be written as

$$(\mathcal{C}^*(\Sigma) \vee \mathcal{V}^*(\Sigma)) \cap \mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}}) = \mathcal{V}^*(\Sigma) \quad (4.106)$$

The complementary property of the subspaces $\mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}})$ and $\mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}})$ gives that the zero modules $\mathcal{W}(\ker F_{\mathbf{r}\mathbf{l}})$ and $\mathcal{W}(\text{Im } F_{\mathbf{r}\mathbf{l}})$ are trivial. In fact, Corollary 3.5 and Theorem 3.8 can be applied (using that the realization of $F_{\mathbf{r}\mathbf{l}}$ is observable).

According Proposition 4 in [1] the invertibility of a proper transfer function is equivalent to that the property that the corresponding $\mathcal{V}^*(\Sigma)$ and $\mathcal{C}^*(\Sigma)$ are complementary subspaces and the columns of $\begin{bmatrix} B \\ D \end{bmatrix}$ are linearly independent, the rows of $[C, D]$ are linearly independent.

In the present situation the complementary property of $\mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}})$ and $\mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}})$ was just proved.

Furthermore, in Remark 3.13 we have checked the left invertibility of $F_{\mathbf{r}}$. Similar argument gives the left-invertibility of $F_{\mathbf{r}\mathbf{l}}$. In fact, if for some vector ξ both $(B + \Pi_l \sigma^{-1} H_k^*) L_0 \xi = 0$ and $L_0' D L_0 \xi = 0$ then the identity $R_0' D = 0$ implies that $D L_0 \xi = 0$. From the first equation we obviously get that $B L_0 \xi \in \text{Im } \Pi_k \subset \text{Im } \Pi$. Using the maximality of R_0 we obtain that $L_0 \xi \in \text{Im } R_0$, i.e. $\xi = 0$. The right invertibility of $F_{\mathbf{r}\mathbf{l}}$ can be proved with obvious modification, concluding the proof of part (iii).

(iv) The Riccati-equation (4.95) using (4.94) can be written as

$$\begin{bmatrix} \Lambda_k' + \beta_k' \alpha_k' & H_k' + \beta_k' R_0' \\ \alpha_k' & R_0' \end{bmatrix} \begin{bmatrix} \sigma' \\ (H_k' + \beta_k' R_0')^* \end{bmatrix} = \begin{bmatrix} -\sigma' (\Lambda_k' + \beta_k' \alpha_k')^* \\ 0 \end{bmatrix} \quad (4.107)$$

Now invoking Lemma 4.2 (for the left multiplication) and Remark 4.20 we get that there exist matrices Y_k', Z_k' and V_k' where the columns of V_k' are in $\text{Im } (\Pi_f)$ such that

$$\Pi_k' Y_k' = \sigma'$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y_k' \\ Z_k' \end{bmatrix} = \begin{bmatrix} -Y_k' (\Lambda_k' + \beta_k' \alpha_k')^* + V_k' \\ -(H_k' + \beta_k' R_0')^* \end{bmatrix} . \quad (4.108)$$

Straightforward calculation gives that

$$\begin{aligned}
& \begin{bmatrix} A & (B + \Pi_k \sigma^{-1} H_k^*) L_0 \\ L'_0 (C + H'_k \sigma'^{-1} \Pi'_k) & L'_0 D L_0 \end{bmatrix} \begin{bmatrix} \Pi_{fzk} & Y'_k \\ L_0^* H_{fzk} & L_0^* Z'_k \end{bmatrix} \\
&= \begin{bmatrix} \Pi_{fzk} (\Lambda_{fzk} + \Gamma H_{fzk}) & -Y'_k (\Lambda'_k + \beta'_k \alpha'_k)^* + V'_k + \Pi_k \sigma^{-1} (H_k^* + \beta_k^* R_0^*) Z'_k \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \Pi_{fzk} (\Lambda_{fzk} + \Gamma H_{fzk}) & -Y'_k (\Lambda'_k + \beta'_k \alpha'_k)^* + \Pi_f \Delta + \Pi_k \sigma^{-1} (H_k^* + \beta_k^* R_0^*) Z'_k \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

where Γ is defined in (3.64) and Δ is defined by $\Pi_f \Delta = V'_k$.

In fact, the term (1, 1) is the same equation as the first term in (3.63). The identity $\Pi'_k \Pi_{fzk} = 0$ implies that the term (2, 1) is essentially identical to the second equation in (3.63).

On the other hand – using that $DL_0 L_0^* = D$, $\sigma' = \Pi'_k Y'_k$ and $L'_0 R_0'^* = 0$ –

$$\begin{aligned}
L'_0 (C + H'_k \sigma'^{-1} \Pi'_k) Y'_k + L'_0 D L_0 L_0^* Z'_k &= -L'_0 (H'_k + \beta'_k R_0')^* + L'_0 H'_k \sigma'^{-1} \Pi'_k Y'_k \\
&= 0,
\end{aligned}$$

and finally – from $BR_0 = \Pi_k \alpha_k$ – we get that

$$\begin{aligned}
AY'_k + (B + \Pi_k \sigma^{-1} H_k^*) L_0 L_0^* Z'_k &= AY'_k + BZ'_k - BR_0 R_0^* Z'_k + \Pi_k \sigma^{-1} H_k^* L_0 L_0^* Z'_k \\
&= -Y'_k (\Lambda'_k + \beta'_k \alpha'_k)^* - \Pi_k (\alpha_k R_0^* - \sigma^{-1} H_k^* L_0 L_0^*) Z'_k + V_k'' \\
&= -Y'_k (\Lambda'_k + \beta'_k \alpha'_k)^* + \Pi_k \sigma^{-1} (H_k + R_0 \beta_k)^* Z'_k + \Pi_f \Delta.
\end{aligned}$$

The identities $\Pi'_k \Pi_{fzk} = 0$, $\Pi'_k Y'_k = \sigma' > 0$ give that the columns of $[\Pi_{fzk}, Y'_k]$ are linearly independent. Furthermore,

$$\begin{aligned}
\text{rank}([\Pi_{fzk}, Y'_k]) &= \dim(\mathcal{V}^*(\Sigma)) + \text{rank}(\Pi'_k) \\
&= \dim(\mathcal{V}^*(\Sigma)) + \dim(\mathcal{V}_{\text{left}}^*(\Sigma) \cap \mathcal{C}_{\text{left}}^*(\Sigma)) \\
&= \dim(\mathcal{V}^*(\Sigma_{\mathbf{r}})) + (n - \dim(\mathcal{V}^*(\Sigma_{\mathbf{r}}) \vee \mathcal{C}^*(\Sigma_{\mathbf{r}}))) \\
&= n - \mathcal{C}^*(\Sigma_{\mathbf{r}}) \\
&= n - \mathcal{C}^*(\Sigma_{\mathbf{r}\mathbf{l}}),
\end{aligned}$$

proving the maximality of $[\Pi_{fzk}, Y'_k]$ using the observation that $\mathcal{C}^*(\Sigma_{\mathbf{r}})$ and $\mathcal{V}^*(\Sigma_{\mathbf{r}})$ are complementary subspaces – and giving that

$$\mathcal{V}^*(\Sigma_{\mathbf{r}\mathbf{l}}) = \text{Im}[\Pi_{fzk}, Y'_k].$$

Using the minimality of the realization $F_{\mathbf{r}\mathbf{l}}$ we obtain that the finite zero matrix of $F_{\mathbf{r}\mathbf{l}}$ is determined by the equation (4.105). This concludes the proof of the theorem. \blacksquare

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